On Last Passage Time in Periodic Environment

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1 Introduction

In 1965, Hammersley and Welsh introduced first passage percolation as a model of fluid flow through a porous medium. The general model in the lattice $\mathbb{Z}^d$ is defined as follows.
For each nearest-neighbor edge e, assign a value τ_e to it, called the weight. The collection of weights is assumed to be i.i.d. with common probability distribution. A path Γ is a finite or infinite sequence of edges $\{e_i\}_{1\leq i\leq n}$ in $\mathbb{Z}^d$ such that e_i and e_{i+1} share exactly one endpoint. In the finite case, the length of any path is the number of edges involved and we define the passage time of Γ to be

$$T(\Gamma) = \sum_{e\in\Gamma} \tau_e$$

Given two points $x, y \in \mathbb{Z}^d$, the first passage time is given as

$$T(x, y) = \inf_{\Gamma} T(\Gamma)$$

where the infimum is over all finite paths Γ that start from x and end at y. Conversely, the last passage time is

$$L(x, y) = \sup_{\Gamma} T(\Gamma)$$

We call

$$\lim_{n\to\infty} \frac{L(0, [nx])}{n} = \mu(x)$$

the time constant. Its existence will be proved by Fekete's Lemma and Kingman's Theorem. The time-constant acts as a law of large numbers for the passage time. In $d = 1$ case, the existence of the time constant can be proved using the law of large numbers described in Appendix.
In this paper we focus on the square lattice in $\mathbb{Z}^2$ lattice and discuss the behavior of the limits of last passage time. We are interested in the time constant or the "average directional speed of fluid-flow"

1.1 Fekete's Lemma and Kingman's Theorem

1.1.1 Fekete's Lemma

Suppose $\{a_n\}_{n\in\mathbb{Z}^+}$ is a real sequence, and suppose also that the sequence satisfies the superadditivity property, i.e.,

$$a_{n+m} \geq a_n + a_m \quad \forall\ n, m \in \mathbb{Z}^+$$

Lemma (Fekete's Lemma)

$$\lim_{n\to\infty} \frac{a_n}{n} = \sup_n \frac{a_n}{n}$$

where the limit could take value with $-\infty$ and ∞
Proof Let $b_n = a_n/n$. Suppose $\sup b_n = \infty$, then as n approaches infinity, b_n tends to infinity and the lemma holds.
Suppose $\sup b_n < \infty$. Notice that $\overline{\lim} b_n$ and $\underline{\lim} b_n$ exist and that

$$\underline{\lim} b_n \leq \overline{\lim} b_n \leq \sup b_n$$

Claim that $b_k \leq \underline{\lim} b_n$ for all $k = 1, 2, ...$
Fix $k \in \mathbb{Z}^+$ such that $k < n$ and $n = pk + q$, then

$$b_n = \frac{a_n}{n} \geq \frac{a_{pk} + a_q}{n} \geq \frac{pa_k}{n} + \frac{a_q}{n} \geq \frac{pk}{n} \cdot \frac{a_k}{k} + \frac{a_q}{n}$$

Since

$$\lim_{n\to\infty} \frac{pk}{n} \cdot \frac{a_k}{k} + \frac{a_q}{n} = \frac{a_k}{k} = b_k$$

We have

$$b_k \leq \underline{\lim} b_n \qquad (k = 1, 2, ...)$$

But then we have

$$\underline{\lim} b_n \geq \sup b_n$$

as $\underline{\lim} b_n$ is an upper bound for b_n.
Since

$$\sup b_n \leq \underline{\lim} b_n \leq \overline{\lim} b_n \leq \sup b_n$$

Then it must be that

$$\underline{\lim} b_n = \overline{\lim} b_n$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim b_n = \sup b_n = \sup_n \frac{a_n}{n}$$

1.1.2 Proposition

For any $x \in \mathbb{Z}^d$, $[nx]$ refers to be vertex where each coordiante is multiplied with n. Consider the expectation of the last passage time from the origin to $[nx]$ for some x, then

Proposition

Let $a_n = E[L(0, [nx])]$ be a sequence, claim that a_n satisfies the superadditivity property, i.e.,

$$E[L(0, [(n+m)x]] \geq E[L(0, [nx])] + E[L(0, [mx])]$$

Proof

Since $L(0, [nx])$ takes the supremum time among all the paths from 0 to $[nx]$, it is clear that

$$L(0, [(n+m)x]) \geq L(0, [nx]) + L([nx], [(n+m)x])$$

It follows that

$$E[L(0, [(n+m)x])] \geq E[L(0, [nx])] + E[L([nx], [(n+m)x])]$$

Since the passage time of edges are assumed to be i.i.d., we can shift the origin to $[nx]$, and thus

$$E[L([nx], [(n+m)x])] = E[L(0, [mx])]$$

Consequently,

$$a_{n+m} = E[L(0, [(n+m)x])] \geq E[L(0, [nx])] + E[L(0, [mx])] = a_n + a_m$$

We have just shown that a_n is a superadditive sequence.

1.1.3 Kingman's Theorem

Since $E[L(0, [nx])]$ is superadditive by above, the Fekete's Lemma says that

$$g(x) = \lim_{n\to\infty} \frac{E[L(0, [nx])]}{n} = \sup_n \frac{E[L(0, [nx])]}{n}$$

exists. The Kingman's Theorem states
Theorem (Kingman's Theorem)

$$\lim_{n\to\infty} \frac{L(0, [nx])}{n} \longrightarrow g(x) \quad a.s. \text{ and in } L^1$$

where $a.s.$ means almost surely and L^1 means absolute difference between two values. Since a.s. law implies convergence in means, we have

$$\lim_{n\to\infty} P\left(\left| \frac{L(0, [nx]) - E[L(0, [nx])]}{n} \right| > \epsilon \right) = 0$$

1.2 Periodic Environment

Periodic Environment is the main focus of our work. In such environment, we define a rectangle, fix its configurations, and extend periodically to all of $\mathbb{Z}^d$. In periodic environment, there is an exact formula for point-to-level limits which will be introduced later. The limit shape is expected to be a polygon so we ask:

1. How close between the limit value and the exact formula?
2. What is the number of facets in the limit shape?
3. How small/big the facets are?

1.3 Max-Plus Algebra

Let A be a $m \times n$ and B be a $n \times p$ real matrix, the usual dot product between A and B is $C = \{c_{ij}\}$ of size $m \times p$ and is given by $c_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$. Max-Plus Algebra, on the other hand, has different operation rules for addition and multiplication. In max-plus algebra, we define

$$a \oplus b = \max(a, b), \qquad a \otimes b = a + b$$

Briefly speaing, usual multiplication is replaced by addition and addition is replaced by max operation The dot product of a $m \times n$ matrix A and a $n \times p$ matrix B, in max-plus algebra, is then characterized as

$$
\begin{aligned}
[A \otimes B]_{ik} &= \bigoplus_{j=1}^{l} a_{ij} \otimes b_{jk} \\
&= \max_{j=1,\ldots,l} \{a_{ij} + b_{jk}\}
\end{aligned}
$$

2 Point-to-Level Limits and Tilt-Velocity Duality

Let x_k be a vertex in the $\mathbb{Z}^d$, a path from x_0 to x_n is denoted as

$$x_{0,n} = (x_k)_{k=0}^{n}$$

For any nearest-neighbor edge (x, x_{e_i}) in $\mathbb{Z}^d$, its potential is defined as

$$w(x, x_{e_i}) = \tau(x, x_{e_i}) + h_i$$

where e_i is the i-th unit vector and $h = (h_1, h_2, ..., h_d)$ is a non-negative vector satisfying certain criterion.

On a finite square lattice with size N, consider Σ to be the set of all Γ that starts from the origin such that $|\Gamma| = N$, or equivalently, all paths with length N. We define

$$G_N(h) = \max_{\Gamma_0, |\Gamma_0|=N} w(\Gamma_0)$$

where $w(\Gamma_0)$ is the sum of all potentials of edges contained in Γ_0

The point-to-level limit is then defined as

$$g_{\text{pl}}(h) = \lim_{N\to\infty} \frac{G_N(h)}{N}$$

Let $\mathcal{R}$ denote all the directions that an edge can take and let $\mathcal{U}$ be the convex hull of $\mathcal{R}$, then

$$g_{\text{pl}}(h) = \sup_{\xi\in\mathcal{U}} \{g_{\text{pp}}(\xi) + h \cdot \xi\}$$

and thus

$$g_{\text{pp}}(\xi) = \inf_{h\in\mathbb{R}^d} \{g_{\text{pl}}(h) - h \cdot \xi\}$$

where

$$g_{\text{pp}}(x) = \lim_{N\to\infty} \frac{L(0, [Nx])}{N}$$

3 The model

We introduce a general setting for model in $\mathbb{Z}^2$ and then show how the periodic environment is generated.

3.1 General Settings

Let x be a vertex on $(\mathbb{Z}^{\geq 0})^2$ and is denoted as (i, j) where i and j are non-negative integers. Construct a directed graph where each x is pointed towards $(i+1, j)$ and $(i, j+1)$. We assign random positive weights from a distribution F to each edge.

Let N be the size of the graph, i.e., the number of vertices in a single row or a single column. Figure 1 shows a sample graph with $N = 5$ and with F be a uniform distribution of $[0, 1)$. In the figure, the upper-most and left-most corner is (0,0) and x increments to the right and y increments downward.

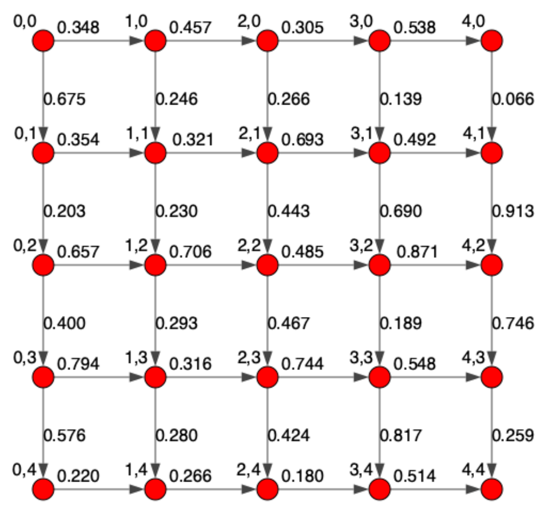


Figure 1: weighted graph with N=5

3.2 Periodic Settings

In periodic environment, we construct a base square lattice with size m first and use a periodic formula to fill a larger lattice with size N following the below procedure

1) Construct a directed square lattice B of size m as defined in Sec. 3.1

2) Extend a directed square lattice G of size $N \geq m$ based on B

3) Assign each edge in B a weight from a distribution F

4) Each vertex $(i, j) \in B$ is identified with

$$P_{i,j} = \{(i + pm, j + qm) \in G : p, q \in \mathbb{Z}^+\}$$

5) Each outgoing edge from $(x, y) \in P_{i,j}$ is assigned with the same weight as the outgoing edge from (i, j) along the same direction

A sample graph with $N = 6, m = 3$ is shown in Figure 2: In the graph above,

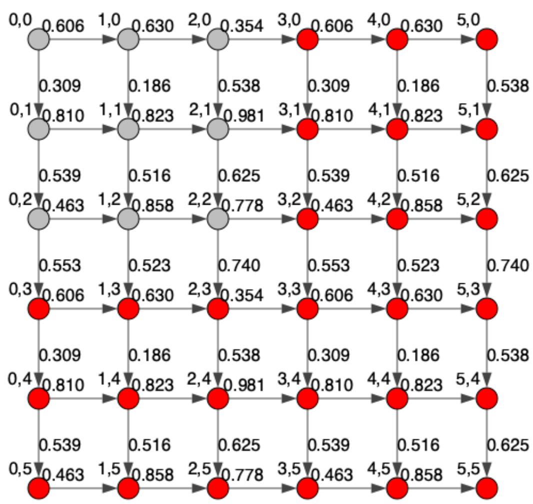


Figure 2: periodic graph with $N = 6, m = 3$

the base square lattice B of size 3 is colored in grey. Each vertex in B has one horizontal edge and one vertical edge. Take $(0, 0)$ for example, it has a horizontal edge to $(1, 0)$ a vertical edge to $(0, 1)$. Now $(0, 0)$ is identified with $(3,0)$ and $(0,3)$. Then we assign the same weight for the edge between $(3,0)$ to $(4,0)$ and for the edge between $(0,3)$ to $(1,3)$ as the weight of the horizontal edge starting from $(0,0)$. We also assign the same weight for the edge between $(3,0)$ to $(3,1)$ and for the edge between $(0,3)$ to $(0,4)$ as the weight of the vertical edge starting from $(0,0)$. Other edges are assigned with weights in a similar fashion.

We now define the weighted adjacency matrix of the periodic graph to be a $m^2 \times m^2$ matrix, denoted as A, where

$$A_{ij} = \begin{cases} w(i,j) & \text{if } i \to j \\ -\infty & \text{otherwise} \end{cases}$$

Notice that in periodic graph, all vertices have degree 2 by the identification step in the periodic formula. One condition for a matrix to be irreducible is that the associated directed graph is strongly connected, meaning any two vertices are reachable to each other by a finite path. The identification process also makes A defined above to be an irreducible matrix.

Theorem

An irreducible matrix A has a unique max-plus eigenvalue $\lambda(A)$

We then show that

$$g_{\text{pl}}(h) = \lambda(A)$$

where A is the weighted adjacency matrix of the periodic graph

For the weighted adjacency matrix of the periodic graph, there is a max-plus eigenvalue λ with an associated eigenvector σ s.t.

$$\max_j [A_{ij} + \sigma_j] = \lambda + \sigma_j, \quad 1 \le i \le N$$

Inductively,

$$\max_{x=x_0,x_1,\ldots,x_n} \left\{ \sum_{k=0}^{n-1} A_{x_k,x_{k+1}} + \sigma_{x_n} \right\} = n\lambda + \sigma_x, \quad 1 \le x \le N$$

The last-passage value can be expressed as

$$G_N(h) = \max_{x_{0:n}} \sum_{k=0}^{n-1} w(x_k, x_{k+1}) = \max_{x=x_0,x_1,\ldots,x_n} \sum_{k=0}^{n-1} A_{x_k,x_{k+1}}$$

Dividing by n gives the limit

$$g_{\text{pl}}(h) = \lim_{n\to\infty} n^{-1} G_N(h) = \lambda$$

4 Algorithms

4.1 Dijkstra's Algorithm

Finding the last passage time with positive weights is mathematically equivalent to finding the first passage time with negative weights. Dijkstra Algorithm is the algorithm that seeks to find the first passage time between each vertex and the source vertex. It repeatedly chooses unvisited nearest estimated vertex, relaxes all edges leaving the vertex, and mark the vertex as visited. We denote G to be the graph, $G.V$ to be the vertex set, w to be the set of edge weights indexed by endpoints, $v.\pi$ be the parent of vertex v.

Algorithm 1: INITIALIZE-SINGLE-SOURCE(G, s)

for $v \in G.V$ **do**
    $v.d = \infty$;
    $v.\pi =$ NULL;
end

Algorithm 2: RELAX(u, v, w)

if $v.d > u.d + w(u, v)$ **then**
    $v.d = u.d + w(u, v)$;
    $v.\pi = u$;
end

Algorithm 3: DIJKSTRA(G,w,s)

INITIALIZE-SINGLE-SOURCE(G,s);
$S = \emptyset$;
$Q = G.V$; **while** $Q \neq \emptyset$ **do**
    $u =$ EXTRACT-MIN(Q);
    $S = S \cup \{u\}$;
    for $v \in G.Adj[u]$ **do**
        RELAX(u, v, w);
    end
end

Running Time
Dijkstra's algorithm implicitly calls INSERT in building Q, DECREASE-KEY in RELAX, and explicitly calls EXTRACT-MIN operations. Each INSERT and DECREASE-KEY operation takes $O(1)$ time while each EXTRACT-MIN operation takes $O(V)$ time where V is the number of vertices. Notice that RELAX is called at most E times where E is the number of edges since the sum of the number of adjacent vertices in the graph is equal to the number of edges in directed graph, then DECREASE-KEY is called at most E times. Now we loop over all the vertices and for each vertex we call EXTRACT-MIN once, the running time is then $O(V^2)$. Together with DECREASE-KEY operation, the total running time is $O(V^2 + E) = O(V^2)$ since $E \leq \frac{V(V-1)}{2} = V^2 - V \leq V^2$ in a directed graph. The running time could be improved using a binary min-heap when the graph is sufficiently sparse. A binary min-heap is a binary tree such that each node has smaller key than the keys of its children. If we implement a min-heap in the algorithm, each EXTRACT-MIN takes $O(\log V)$ time and the running time for the EXTRACT-MIN operation considering the loop through all the vertices is then $O(V \log V)$. Each DECREASE-KEY now takes $O(\log V)$ time. The time to build the heap is $O(V)$ so the total running time for the improved algorithm is then $O((V + E) \log V)$. Since all vertices are reachable from the source, the running time is therefore $O(E \log V)$.

4.2 Karp's Algorithm

Karp's Algorithm is one of the algorithms that serve to solve for the eigenvalue problem in Max-Plus Algebra, $i.e.\lambda$ such that $A \otimes v = \lambda \otimes v$.

Algorithm 4: KARP'S ALGORITHM

1. Choose $j \in \underline{n}$ and set $x(0) = e_j$;
2. Compute $x(k) = A \otimes x(k-1)$ for $k = 1, ..., n$;
3. Compute $\lambda = \max_{i=1,...,n} \min_{k=0,...,n-1} \frac{x_i(n) - x_i(k)}{n-k}$

where A is the weighted adjacency matrix, $\underline{n}$ represents $1, ..., n$ and $x_i(m)$ refers to the i-th element of $x(m)$.

Example Let

$$A = \begin{pmatrix} \epsilon & 3 & \epsilon & 1 \\ 2 & \epsilon & 1 & \epsilon \\ 1 & 2 & 2 & \epsilon \\ \epsilon & \epsilon & 1 & \epsilon \end{pmatrix}$$

Apply Karp's Algorithm with $j = 1$,and consider $x(0) = e_1 = (0, \epsilon, \epsilon, \epsilon)^T$. Since $n = 4$, there are four iterations and we get

$$x(1) = \begin{pmatrix} \epsilon \\ 2 \\ 1 \\ \epsilon \end{pmatrix}, \quad x(2) = \begin{pmatrix} 5 \\ 2 \\ 4 \\ 2 \end{pmatrix}, \quad x(3) = \begin{pmatrix} 5 \\ 7 \\ 6 \\ 5 \end{pmatrix}, \quad x(4) = \begin{pmatrix} 10 \\ 7 \\ 9 \\ 7 \end{pmatrix}$$

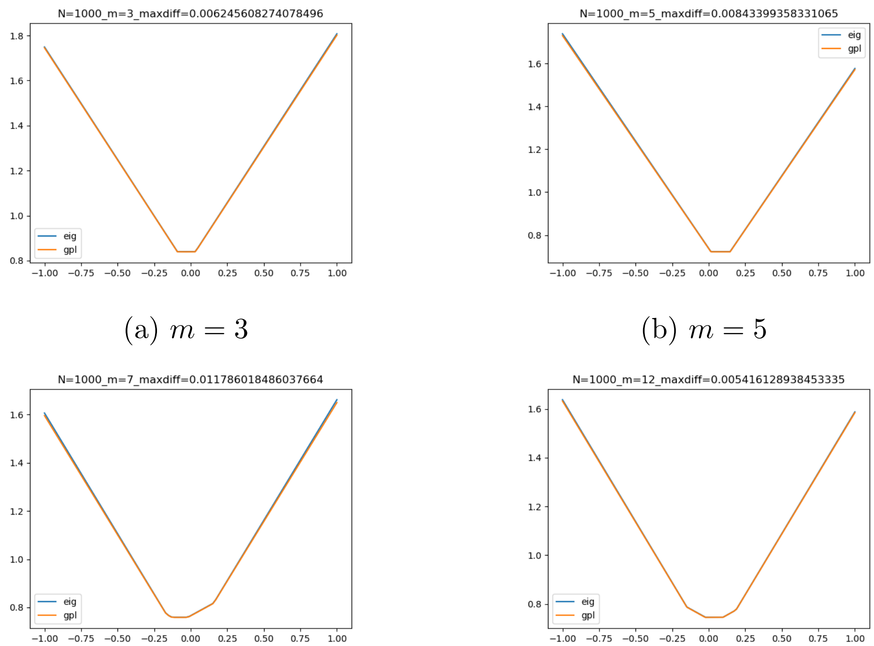
The mininum values over k are $\frac{5}{2}, 0, \frac{5}{2}, 2$ respetively. Then the final result after taking maximum is $\frac{5}{2}$. The max-plus eigenvalue is numerically equivalent to the point-to-level limits.

Running Time

Each element of $x(k)$ is calculated from taking maximum of n elements which takes $O(n)$ time, then calculating each $x(k)$ takes $O(n^2)$ time. There are n such $x(k)$ in total, the running time for the dot product of A and v in max-plus algebra is thus $O(n^3)$. The eigenvalue is calculated by taking the maximum of the minimum of arrays of elements. Since the max process and the min process each takes $O(n)$ time, the total running time of Karp's algorithm is $O(n^3 + n^2) = O(n^3) = O(m^6)$ where m is the size of the base square lattice.

5 Results

We perform simulations for $N = 1000, m = 3, 4, 5, ..., 15$ and the results of four chosen periodicities are shown in Figure 3. Increased number of facets in the shape could be observed with the increase in periodicities of the graph. When the period is $3 - 6$, there are roughly 3 facets; when the period is 7-11, there are roughly 4 facets; and when the period is at least 12, the number of facets becomes at least 5. The observed pattern showed a moderate to fast increase in number of facets with most facet changes within $h = -0.25$ to $h = 0.25$



(a) $m = 3$ (b) $m = 5$

(c) $m = 7$ (d) $m = 12$

Figure 3: plots of gpl and eigenvalues with different periodicities

6 Appendix

6.1 Weak Law of Large Numbers

Suppose $\{X_i\}$ are i.i.d. where X_i has cdf f and let $S_n = X_1 + X_2 + \cdots + X_n$ to be the sum of the first n terms in the sequence. The Weak Law of Large Numbers states

Proposition

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - E[X_1]\right| > \epsilon\right) = 0$$

Proof

Let μ, σ be the mean and standard deviation of S_n/n. Let μ_x, σ_x be the mean and standard deviation of X_i. By Central Limit Theorem,

$$\mu = \mu_x, \qquad \sigma = \frac{\sigma_x}{\sqrt{n}}$$

Chebyshev's inequality states

$$P\left(\left|\frac{S_n}{n} - E[X_1]\right| > k\sigma\right) \leq \frac{\sigma^2}{k^2} = \frac{\sigma_x^2}{nk^2}$$

Take $k = \frac{\epsilon}{\sigma}$, then

$$P\left(\left|\frac{S_n}{n} - E[X_1]\right| > \epsilon\right) \leq \frac{\sigma^4}{n\epsilon^4}$$

Therefore,

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - E[X_1]\right| > \epsilon\right) = 0$$

7 Bibliography

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