

Asymptotic Solution for the One Dimensional Euler Equations for Isentropic Flow in a Variable Area Duct

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The objective of this paper is to derive an asymptotic solution to the one-dimensional Euler equations for isentropic flow through ducts with slowly varying area. The homogeneous (or constant-area) problem is generally handled using Riemann's method of characteristics. We solve the variable-area problem by an asymptotic expansion about this homogeneous solution. A length scale characterizing the area variation is introduced and an asymptotic power series in increasing powers of the inverse of the length scale ε are constructed for the unknowns. The problem reduces to solving the homogeneous Euler equation, and coupled linear PDEs for successive correction terms in the asymptotic series. An integration methodology is also presented for simple wave regions. As an illustration, we obtain closed-form analytical expressions for the first order perturbation terms in the case of an exponential duct for a sample simple wave. Nonlinear distortion of the wavefront is captured accurately in the analytical solution, as verified by comparison with numerical results from CLAWPACK, a finite-volume simulation package for conservation laws. Other issues such as asymptoticity and convergence of the series are discussed.

Nomenclature

u	=	velocity of fluid
a	=	speed of sound
p	=	pressure
ρ	=	density
\mathcal{J}^\pm	=	Riemann invariants
γ	=	ratio of specific heats

I. Introduction

WAVE motion is a broad subject studied under almost all engineering disciplines and is responsible for many physical phenomena. We are concerned with the propagation of waves from the gas dynamic point of view. The flow of a non-viscous, non-heat-conducting, compressible gas is described by the Euler equations.

The Euler equations come under a class of hyperbolic partial differential equations which have in general, waves as solutions. They represent disturbances in fluid properties propagating spatially with time. The waves governed by these equations are themselves termed hyperbolic waves and since we ignore viscous dissipation, they are also called non-dispersive. These equations are partial differential equations (PDEs), and are reasonably well understood in one dimension, where they are usually used to model wave propagation through ducts. The solution for the Euler equations for one-dimensional duct-flow with no heat transfer, area variation or other source terms (henceforth termed the homogeneous Euler equations) in the case of a plane wave, was given by Earnshaw¹ using an analogous solution in the linear theory. The deeper interpretation using the method of characteristics for general boundary conditions was given by Riemann¹. However, when inhomogeneities like a changing duct area or heat addition on the gas are present, they introduce source terms into the equation which makes their solution difficult. These inhomogeneities play an important role in determining wave propagation and affect wave speed, strength, shock formation, etc. considerably.

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Certain combinations of initial and boundary conditions generate special kinds of waves called simple waves where the problem is analytically tractable. Theoretically, even the general initial value problem which has interacting and simple wave regions, can be solved using a combination of Riemann’s hodograph transformation and the simple wave solution, but this does not find much use in practice. The method of characteristics on the other hand, lends itself easily to numerical approaches to solve general initial/boundary value problems.

When additional geometrical constraints such as a variable duct area or spherical and cylindrical symmetry and are imposed on a problem, source terms are introduced and waves no longer remain ‘simple’. The same is the case when waves propagate into nonuniform regions; i.e., when there are gradients in pressure, or entropy. Approximate, weakly nonlinear solutions, similarity solutions for special problems and wavefront expansions are some methods used to handle inhomogeneities¹. For example, Taylor’s point-blast problem of a strong shock wave produced by an explosion at a point has a similarity solution. The wave-front expansion technique can be used to predict the evolution of the slope of the leading edge of a wavefront for various kinds of inhomogeneities^{1,2,3}. A collection of many classic papers in gas dynamics including those by Earnshaw, Riemann, Rankine and Taylor can be found in Ref. 4.

In this paper, we will solve the Euler equations for isentropic flow in regions of flow where simple waves were present prior to the introduction of a source term due to geometry: variation of duct area. Our solution is inspired by the *method of multiple scales* used in linear theory^{1,5,6}. For example, we can analyze the propagation of shocks governed by the linearized equations in ducts with slowly varying area by perturbing the solution about the known homogeneous case. Similarly, we introduce a length scale and then construct asymptotic expansions for the unknowns, thereby decomposing a nonhomogeneous nonlinear problem into the homogeneous Euler equation for the leading order term and linear PDEs for higher-order correction terms in the asymptotic series. The technique is easily extended to the cases of cylindrical or spherical symmetry where source terms having similar form are introduced.

An analytical solution of this form is useful in many ways: it might provide physical insight into the problem; it provides a simple method to quickly, but approximately quantify the behavior of a wave without having to use numerical simulation packages; it can also be used to benchmark new simulation codes. At the outset, it was also likely that it would capture effects like reflection of the wavefront due to area change and nonlinear distortion since perturbing area alone places no smoothness or magnitude restriction on flow quantities, or focus on any particular region of the flow. For example, the wavefront expansion technique, while it can handle a wide variety of inhomogeneities, tracks the leading edge of the wavefront alone and it is difficult to analyze how trailing parts of the wavefront are affected by the inhomogeneity.

This technique might find application in finite volume simulations. In first-order Godunov type methods, the interval is divided into grid cells and Riemann problems; i.e., discontinuous initial data across grid-cell boundaries, for the homogenous equations are solved. Inhomogeneities are included in the form of source terms and are usually approximated with first order accuracy. An asymptotic solution might be used to solve the appropriate variable-area Riemann problem instead.

In section II, we write the governing equations for one-dimensional isentropic flow in ducts with varying area. In section III, we analyze the homogeneous equations in greater detail laying the foundation for our solution in the section that follows. Section IV describes the asymptotic expansion technique used and gives an integration methodology for the case of simple waves. In the following sections, a model problem is analyzed and compared with numerical results from simulation software. Asymptoticness and convergence of the series are also discussed.

II. Governing Equations

The method of characteristics is used to obtain solutions to linear and in some cases to nonlinear partial differential equations. This involves finding characteristic curves along which the PDE becomes an ODE. Riemann’s classical solution to the constant area problem involves manipulating Euler’s equations (1a-c) into their characteristic form^{1,6}. The governing equations are:

Continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = -\frac{\rho u}{A} \frac{dA(x)}{dx} \quad (1a)$$

Momentum,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{dp}{dx} = 0 \quad (1b)$$

Energy,

$$\frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} = \frac{\dot{q}}{T} \quad (1c)$$

where ρ is the density of the gas, $A(x)$ is the cross-sectional area of the duct, p is the pressure, s is the entropy, T is the temperature and \dot{q} is the specific rate of heat transfer. The gas is assumed to be perfect (polytropic); i.e., the ideal gas equation applies and the specific heats are constant. In addition, we assume that no heat is added and the flow is locally reversible, which implies it is isentropic and hence the pressure p of a fluid particle is a function of ρ alone. We use the standard definition of the isentropic speed of sound a :

$$a^2 = \left. \frac{\partial p}{\partial \rho} \right|_{s=\text{const}}$$

Then, Eqs. (1a-c) can be reduced to a pair of coupled, nonlinear differential equations for u and a that are in characteristic form i.e., reduce to ODEs along $X_+(t)$, $X_-(t)$ and $X(t)$, the characteristic curves.

$$\begin{aligned} \frac{\partial}{\partial t} \left(u + \frac{2a}{\gamma-1} \right) + (u+a) \frac{\partial}{\partial x} \left(u + \frac{2a}{\gamma-1} \right) &= -\frac{ua}{A} \frac{dA}{dx} & \frac{dX_+}{dt} &= u+a \\ \frac{\partial}{\partial t} \left(u - \frac{2a}{\gamma-1} \right) + (u-a) \frac{\partial}{\partial x} \left(u - \frac{2a}{\gamma-1} \right) &= \frac{ua}{A} \frac{dA}{dx} & \frac{dX_-}{dt} &= u-a \\ \frac{\partial s}{\partial t} + u \frac{\partial s}{\partial x} &= 0 & \frac{dX}{dt} &= u \end{aligned} \quad (2a,b,c)$$

The quantities $u \pm 2a/(\gamma-1)$ are called Riemann invariants and are usually denoted by J^\pm . The operators $\partial_t + (u \pm a)\partial_x$ are material derivatives; they represent the change in J^\pm along their respective characteristic curves and are represented by D^\pm/Dt . The characteristics may be interpreted as ‘carrying’ disturbances in pressure or density.

We will first focus on the homogeneous case, where the source terms on the right-hand-side (R.H.S) of Eqs. (2a, b) are zero implying that the Riemann invariants J^\pm are constant along them – hence the term ‘invariant’. Their value at any point on a characteristic curve is the same as the initial/boundary value and the entire flow can be determined by step-wise integration along characteristics once these are specified^{1,7-9}. Generally, the equations imply the existence of three waves: two waves preserving the Riemann invariants traveling with velocities $u \pm a$ and an entropy wave traveling with particle velocity u along which entropy is constant. These three waves interact in a complicated manner as they propagate, and are together known as a *compound wave*. However, if the fluid is uniformly at rest at $t = 0$, it follows from Eq. (2c) that the entropy is constant (if shocks do not occur) at all times throughout the length of the duct. Such a flow is termed *homentropic*⁷. We may then write:

$$p(x,t) = k\rho(x,t)^\gamma \quad (2d)$$

where k is a constant for the whole flow. Moreover, one of the Riemann invariants is constant and there will be a single disturbance propagating in one direction. This is termed as a *simple wave*. Equivalently, those portions of the t - x plane bounded by uniform regions are called simple-wave regions^{1,6,7}.

III. Problem formulation

In the special case of homentropic flow described above, Eqs. (2a-c) can be written succinctly as:

$$\frac{D^\pm}{Dt} J^\pm = 0 \quad (3a,b)$$

A canonical problem is that of a piston moving in a duct generating waves, with the boundary condition being specified on the wall of the piston. The case when the piston withdraws with a constant velocity, creating an expansion wave is a standard problem in gas dynamics and the solution has an especially simple analytical form^{1,9}. In the interest of numerical simulation in our case, however, the boundary velocity is specified at $u(x=0,t) = f(t)$ – the so-called ‘loudspeaker’ boundary condition in gas dynamics. A piston problem would involve handling a moving boundary when simulated, which is a little more difficult to handle.

Equation (3b) represents backward traveling waves with velocity $u - a$. These will be referred to as J^- waves and the characteristic lines in the $t-x$ plane corresponding to these waves will be referred to as C^- characteristics. These waves originate at various points on the x -axis and travel backwards (with negative slope) towards the t -axis. The flow is at rest initially $u = 0, a = a_0$ and the value of J^- over the entire $t-x$ plane is equal to its value on the x -axis. Therefore,

$$\frac{2a}{\gamma-1} - u = \frac{2a_0}{\gamma-1} \quad \forall x,t \quad (4)$$

We now have a relationship between u and a , and have reduced the problem to solving just one equation. J^+ is constant along its own characteristic lines C^+ , and since J^- is constant everywhere, u and a are individually constant along C^+ characteristics implying that they are straight lines. A general C^+ line originates on the t -axis at $(0, \tau)$ and its equation can be written as:

$$x = \left\{ a_0 + \frac{\gamma+1}{2} f(\tau) \right\} (t - \tau) \quad (5)$$

Given a point in the $t-x$ plane, the point of origin of the characteristic passing through it is given implicitly by Eq. (5) and the velocity at that point is simply $u = f(\tau)$ with a being determined from Eq. (4). (See Fig. 1).

We see from Eq. (5) that when $f(t)$ (in absolute value) exceeds $2a_0/(\gamma+1)$ – the *sonic velocity* –, the C^+ lines

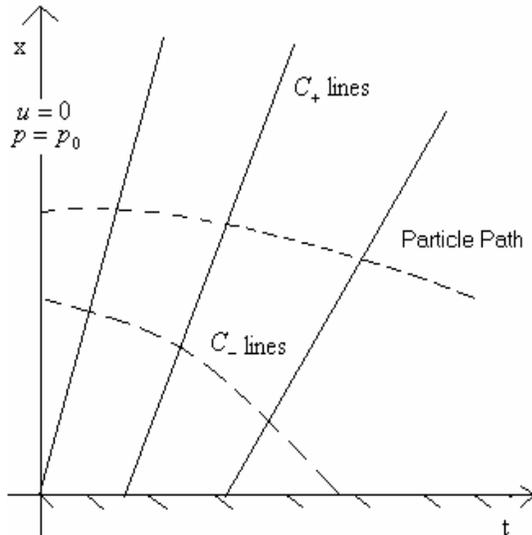


Figure 1. General boundary velocity. The figure shows the characteristics and the particle paths for a generic simple wave.

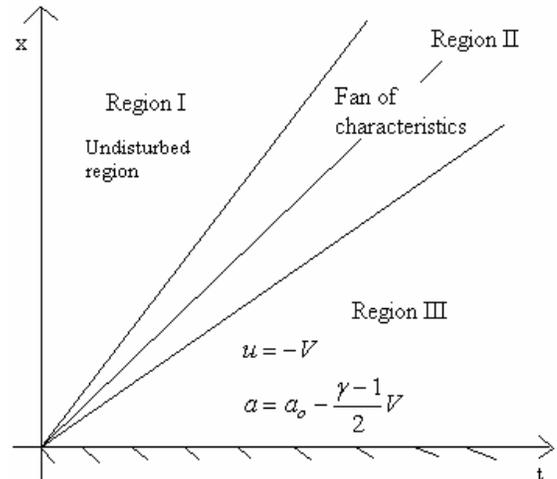


Figure 2. Constant velocity. In this case, there is a singularity at the origin and all velocity values in the range 0 to $-V$ are taken at this point. Notice the ‘fan’ structure: this is called a centered simple wave.

have negative slope and the waves travel in the backward direction. When $f(t) > -2a_0/(\gamma-1)$, the *escape velocity*, a and p become zero. To avoid such unphysical situations, we assume that $f(t)$ and its first derivative are continuous and bounded. If at any point, $\dot{f}(t) > 0$, then its C^+ characteristic will intersect with others and the point of intersection will have two different values of velocity. This is called a shock and flow quantities are typically discontinuous here. To interpret this phenomenon physically, the more generally valid integral form of the conservation laws corresponding to Eqs. (1a-c) must be used. Usually, a ‘weak solution’ is attempted by using shock-conditions. To avoid this additional complication of shock formation, we assume $f(t) \leq 0$ and $\dot{f}(t) \leq 0$. The simple wave produced under such assumptions is called an expansion wave.

We next consider the case where $f(t) = -V$ for $t > 0$, where Eq. (5) and the entire problem have simple closed form solutions. At time $t = 0$, the fluid is at rest and at $t = 0^+$, the velocity at the boundary instantly takes the value $-V$ and remains there. This, as can be inferred from Figs. 1 and 2, is the limiting case of infinite acceleration at the origin and all the C^+ characteristics emerge from the origin forming a characteristic ‘fan’^{1,6}. The particles at the origin instantaneously take all the values from 0 to $-V$ and each such velocity results in one member of the fan. The two limiting characteristic lines of the fan (Fig. 2) divide the $t-x$ plane into three regions and the solution can be written as:

$$\begin{aligned}
 u = 0, a = a_0 & & \frac{x}{t} \geq a_0 \\
 u = \frac{2}{(\gamma+1)} \left(\frac{x}{t} - a_0 \right), a = \frac{\gamma-1}{\gamma+1} \frac{x}{t} + \frac{2}{\gamma+1} a_0 & \text{ for } a_0 - \frac{\gamma+1}{2} V \leq \frac{x}{t} \leq a_0 \\
 u = -V, a = a_0 - \frac{\gamma-1}{2} V & & \frac{x}{t} \leq a_0 - \frac{\gamma+1}{2} V
 \end{aligned} \tag{6}$$

The solution represents a plane expansion wave front traveling right bringing the whole duct into the flow. The C^+ line $x = a_0 t$ is called the ‘first signal’ and is the fastest traveling characteristic. Points to the left of the first signal represent regions as yet undisturbed by the wave. The region through which the wave has passed is uniform with velocity $-V$. This expansion front ‘relaxes’ as it propagates towards the right: this effect is the expansion wave counterpart of nonlinear steepening in compressive waves. That is, it starts off at $t = 0$ with an infinite slope and this slope relaxes to zero as the $t \rightarrow \infty$. We can see this in Fig. 2, where the width of the characteristic fan increases linearly with time. Also, differentiating u as given by Eq. (6) with respect to x , i.e., the slope of the expansion fan at any instant of time, shows clearly that it goes to zero as $t \rightarrow \infty$.

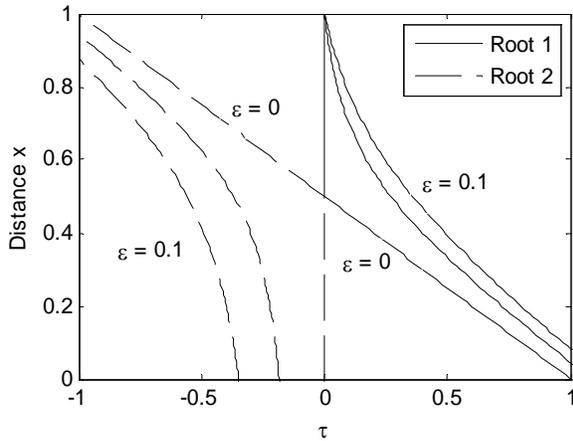


Figure 3. Behavior of the roots of Eq. (5) for fixed V and t . Notice the limiting behavior at $\varepsilon = 0$. The branch of root 1 where it is identically zero represents the characteristics fan

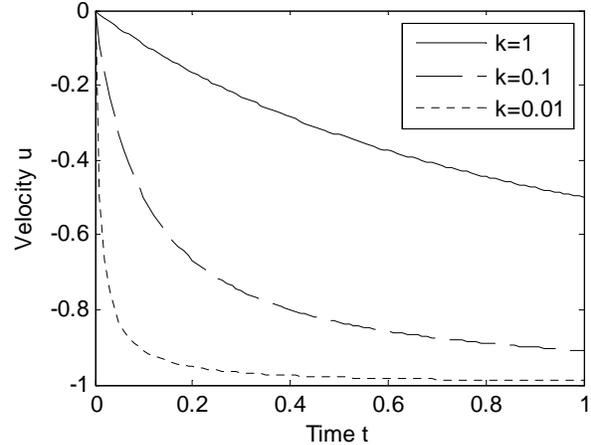


Figure 4. $F(t)$ vs. t for different values of k . As k approaches zero, the profile matches the constant velocity case.

The structure of the characteristics is not immediately apparent from Eq. (5). To see this, we must approach this discontinuous case as a limit of a continuous case. We consider one such case where $u(0, t) = -Vt/(t+k)$ for $t \geq 0$ as shown in Fig. 4. The velocity starts at zero and asymptotically reaches velocity $-V$. The acceleration starts at a finite

value, is negative for all other finite times and approaches zero as $t \rightarrow \infty$. Now, Eq. (5) is quadratic in τ for any point (x, t) :

$$(a_0 - \frac{\gamma+1}{2}V)\tau^2 - \left\{ (a_0 - \frac{\gamma+1}{2}V)t - (a_0k + x) \right\} \tau - k(a_0t - x) = 0 \quad (7)$$

The behavior of the roots as $k \rightarrow 0$ must be investigated. To this end, at appropriately chosen values of t , V and a_0 , we plot the variation of τ with x in Fig. 3, where it can be seen that the one root is negative and must be discarded since the flow is uniformly at rest for $t < 0$. Then, the solution for τ is:

$$\tau = \begin{cases} 0 \\ \{x - (a_0 - \bar{V})t\} / (a_0 - \bar{V}) \end{cases} \quad \text{for} \quad \begin{cases} (a_0 - \bar{V})t \leq x \leq a_0t \\ 0 \leq x \leq (a_0 - \bar{V})t \end{cases} \quad (8)$$

where $\bar{V} = V(\gamma+1)/2$. This confirms the structure of the characteristics shown in Fig. 2. Physically, the initial acceleration increases as k approaches zero and the characteristics crowd closer and closer together until they emerge from the origin as the acceleration approaches infinity. In the uniform region, the C^+ are parallel with slope $a_0 - \bar{V}$. When V takes the sonic velocity, i.e., $\bar{V} = a_0$, the square term in Eq. (6) drops out and there remains only one root for τ . The solution then turns out to be:

$$\begin{aligned} u &= \frac{2}{(\gamma+1)} \left(\frac{x + a_0k}{t+k} - a_0 \right) \\ a &= \frac{\gamma-1}{\gamma+1} \left(\frac{x + a_0k}{t+k} \right) + \frac{2}{\gamma+1} a_0 \end{aligned} \quad \text{for} \quad 0 \leq x \leq a_0t \quad (9)$$

Certainly, these equations look similar to those in Eq. (6) and are identical for $k = 0$. It follows that this case is exactly the constant velocity case, but with a coordinate translation (See Fig. 5).

$$\begin{aligned} x' &= x + a_0k \\ t' &= t + k \end{aligned} \quad (10)$$

In the x, t plane, the characteristics form a ‘fan’ – also called a centered simple wave – just like the constant velocity case. Closed-form expressions for the perturbation terms are simpler to obtain when inhomogeneities are introduced later. There is also the advantage that the wavefront encompasses the entire region of flow; there is no uninteresting uniform-flow region.

IV. Asymptotic expansion of the Euler equations

An asymptotic expansion is a functional series which approximates a solution to an algebraic or differential equation⁵. They are written out in terms of a small parameter ε , which is usually an introduced perturbation like a small nonlinear source term. Asymptotic expansions are sometimes written out as series in powers of ε , in which case they are called asymptotic power series:

$$f(x) \approx \sum_{n=0}^N a_n(x) \varepsilon^n$$

Asymptotic series need not converge, but must approximate the solution well for small values of ε . The leading order term must be equal to the solution for $\varepsilon = 0$ and higher order terms must be corrections of decreasing order of magnitude. One usually constructs them as perturbations *about* a solution (root) of a differential (algebraic) equation.

To construct an asymptotic expansion for u and a , we assume the area of the duct is ‘slowly’ varying over x . That is, it is assumed to be characterized by a large length scale (L) in comparison to the coordinate x over which the area variation takes place. The inverse of L is small and is called ε . Usually, a length scale is defined as:

$$\frac{L}{A_0} \frac{dA(x)}{dx} \Big|_{x=x_0} = 1 \quad \text{or} \quad \frac{1}{A} \frac{dA(x)}{dx} \Big|_{x=x_0} = \varepsilon \quad (12)$$

Following this, the area of the duct is then written as $A(x) = G(\varepsilon x)$ or more precisely, $A'(x)/A(x) = \varepsilon f(x)$, where $f(x) \square O(1)$. The source term corresponding to area variation on the R.H.S of Eqs. (2a,b) then becomes:

$$\frac{\varepsilon}{G(\varepsilon x)} \frac{dG(\varepsilon x)}{d(\varepsilon x)} \quad (13)$$

Then we express the velocity and speed of sound as power series in ε .

$$\begin{aligned} u &= u_* + \varepsilon u_1 + \varepsilon^2 u_2 + \dots \\ a &= a_* + \varepsilon a_1 + \varepsilon^2 a_2 + \dots \end{aligned} \quad (14)$$

On substitution of Eq. (14) in Eq. (2) and grouping terms in powers of ε , we obtain:

$$\begin{aligned} &\varepsilon^0 \left\{ \frac{\partial}{\partial t} \left(u_* \pm \frac{2a_*}{\gamma-1} \right) + (u_* \pm a_*) \frac{\partial}{\partial x} \left(u_* \pm \frac{2a_*}{\gamma-1} \right) \right\} + \\ &\varepsilon^1 \left\{ \frac{\partial}{\partial t} \left(u_1 \pm \frac{2a_1}{\gamma-1} \right) + (u_* \pm a_*) \frac{\partial}{\partial x} \left(u_1 \pm \frac{2a_1}{\gamma-1} \right) + (u_1 \pm a_1) \frac{\partial}{\partial x} \left(u_* \pm \frac{2a_*}{\gamma-1} \right) \right\} + \\ &\varepsilon^2 \left\{ \frac{\partial}{\partial t} \left(u_2 \pm \frac{2a_2}{\gamma-1} \right) + (u_* \pm a_*) \frac{\partial}{\partial x} \left(u_2 \pm \frac{2a_2}{\gamma-1} \right) + (u_1 \pm a_1) \frac{\partial}{\partial x} \left(u_1 \pm \frac{2a_1}{\gamma-1} \right) + \frac{\partial}{\partial x} (u_2 \pm a_2) \left(u_* \pm \frac{2a_*}{\gamma-1} \right) \right\} \\ &+ \dots = \mp \frac{\varepsilon dG(\varepsilon x)}{G(\varepsilon x)d(\varepsilon x)} \left[u_* a_* + \varepsilon^1 (u_* a_1 + u_1 a_*) + \dots \right] \end{aligned} \quad (15)$$

On comparing coefficients of ε , we obtain coupled pairs of linear differential equations for the terms u_n and a_n only in terms of known lower order terms. Note that the coefficient of ε^0 on the R.H.S of the above equation is zero and the equation for the leading order term in the expansion is just the same as Eq. (2) for a constant area duct. This meets the requirement that the leading order term in the asymptotic expansion must approximate the solution for very small ε . That is, the expansion is constructed about the case where ε is zero – the constant area case.

The boundary conditions of the problem are satisfied using the leading order terms and all higher-order terms are assumed to vanish on the boundaries. A more concise, but consistent notation is adopted: the terms $u_* \pm 2a_*/(\gamma-1)$ are written as J_n^\pm . While J_n^\pm are certainly not invariant along the characteristics, we still call them n^{th} order Riemann invariants, although the term ‘Riemann variables’

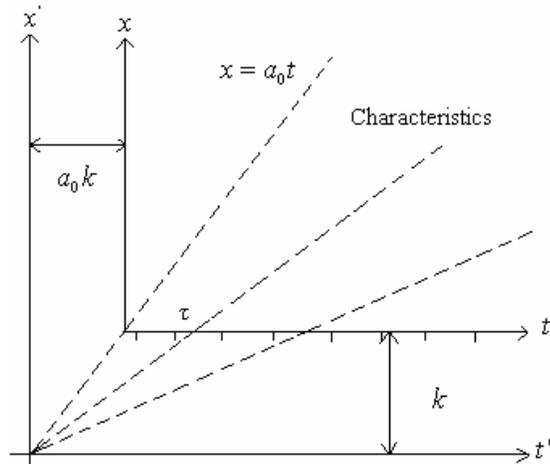


Figure 5. $t-x$ plane for asymptotic boundary. The figure also shows the origin shift. The reason the choice of the sonic velocity simplifies the problem can be seen here. It is required for the characteristics to meet at a single point and span the entire region of interest.

might be more appropriate, as suggested in Ref. 1. The material derivative operator corresponds to the derivative along the characteristic curves of the leading order terms. We will continue to call these curves C^+ and C^- . The equations for a general pair of terms in the series may then be written as:

$$\frac{D^\pm}{Dt} J_n^\pm + (u_n \pm a_n) \frac{\partial}{\partial x} J_n^\pm + \sum_{i=1}^{n-1} (u_i \pm a_i) \frac{\partial}{\partial x} J_{n-i}^\pm = \mp \frac{G'(\varepsilon x)}{G(\varepsilon x)} \sum_{i=0}^{n-1} u_i a_{n-1-i} \quad (16a,b)$$

It is clear that we may integrate all these equations along the characteristics of the leading order term itself. The terms $u_n \pm a_n$ can be written in terms of the invariants as $\lambda_1 J_n^+ + \lambda_2 J_n^-$ and the rest of the terms in Eqs. (16a, b) are known and serve as source terms.

In general, the ease of solution depends upon the boundary conditions since they determine the behavior of the characteristics. If the characteristics and the source terms behave well enough – for example, simple wave problems such as those described in section II – solutions to Eq. (16a, b) may be found. Then, for a right-traveling simple wave in a constant area duct, J_n^+ is constant through out the $t-x$ plane, all its derivatives are identically zero and the second term on the L.H.S of Eq. (12b) drops out.

$$\frac{D^-}{Dt} J_n^- = - \sum_{i=1}^{n-1} (u_i - a_i) \frac{\partial}{\partial x} J_{n-i}^- + \frac{G'(\varepsilon x)}{G(\varepsilon x)} \sum_{i=0}^{n-1} u_i a_{n-1-i} \quad (17)$$

The terms on the R.H.S are just source terms and Eq. (17) can be integrated along its characteristic $X_-(t)$ beginning on the x -axis, or equivalently, a point on the first signal $(a_0 \tau, \tau)$ to a general point on it.

$$J_n^-(t') \Big|_\tau^t = \int_\tau^t H(X_-(t'), t') dt' \quad (18)$$

This value is substituted into the equation for J_n^+ :

$$\frac{D^+}{Dt} J_n^+ + \lambda_1 J_n^+ \frac{\partial}{\partial x} J_n^+ = -\lambda_2 J_n^- \frac{\partial}{\partial x} J_n^+ - \sum_{i=1}^{n-1} (u_i + a_i) \frac{\partial}{\partial x} J_{n-i}^+ - \frac{G'(\varepsilon x)}{G(\varepsilon x)} \sum_{i=0}^{n-1} u_i a_{n-1-i} \quad (19)$$

This is an ordinary differential equation along its characteristic and integrating factors maybe found in some cases and the problem may be reduced to finding an integral. The first terms are found and then substituted into the equations for the second term and the process can be continued to obtain higher-order terms.

V. Solution for a special case

This integration procedure will be illustrated for the first order term for the special boundary velocity $u(0, t) = F(t) = -Vt/(t+k)$, that was discussed previously. When V is chosen to be the sonic velocity, the solution has the form in Eq. (9). Indeed, this solution is nearly identical in any region of flow containing centered simple waves; i.e., those regions covered by a fan of characteristics. Now, the area of the duct is assumed to vary exponentially where a positive exponent represents an expanding duct area and a negative exponent represents a converging duct. For such exponential ducts, we may write:

$$G(\pm \varepsilon x) = A_o \exp(\pm \varepsilon x) \Rightarrow \frac{\pm \varepsilon}{G(\varepsilon x)} \frac{dG(\varepsilon x)}{d(\varepsilon x)} = \pm \varepsilon \quad (20)$$

For notational convenience, we use x, t now to represent the translated co-ordinates, but during the subsequent discussion and results sections, x and t represent ‘real’ coordinates. It is clear that the partial derivatives of the leading order Riemann invariants, in the transformed co-ordinates given by Eq. (10), are:

$$\frac{\partial}{\partial x} J_n^+ = \frac{4}{\gamma+1} \frac{1}{t}, \quad \frac{\partial}{\partial x} J_n^- = 0 \quad (21)$$

The equations for the first order terms for an exponentially expanding duct can then be written as:

$$\begin{aligned}\frac{D^-}{Dt} J_1^- &= u_* a_* \\ \frac{D^+}{Dt} J_1^+ + (\lambda_1 J_1^+ + \lambda_2 J_1^-) \frac{\partial}{\partial x} J_*^+ &= -u_* a_*\end{aligned}\quad (22a,b)$$

The initial and boundary conditions are satisfied by the leading order solution and hence the higher order terms must vanish on the boundaries: $u_n(a_0(t+k), (t+k)) = 0$; $a_n(x, 0) = 0$ $u_n(a_0 k, k) = 0$; $a_n(a_0 k, k) = 0$. (See Fig. 5)

A. Solution for J_1^-

To integrate Eq. (22a) for J_1^- , the equation for its characteristic C^- , must first be solved. From Eq. (2b) and Eq. (9), we obtain:

$$\frac{dX_-(t)}{dt} = \left(\frac{3-\gamma}{\gamma+1}\right) \frac{x}{t} - \left(\frac{4a_o}{\gamma+1}\right) \quad (23)$$

Any C^- line in the expansion fan originates from the first signal at some $(a_0 \tau, \tau)$. Equation (23) can be solved easily by multiplication with the integrating factor t^{-p} , where the constant p is defined below.

$$\frac{x}{t} = p \left(\frac{t}{\tau}\right)^{p-1} + qa_0 \quad \text{and} \quad \left(\frac{t}{\tau}\right)^{p-1} = \frac{a_*}{a_0}; \quad \text{where } p = \frac{3-\gamma}{\gamma+1}, q = \frac{2}{\gamma-1} \quad (24a,b)$$

The source term in Eq. (22a) is known since the constant area solution is known; the value of x/t is from Eq. (24a); the initial condition is $J_1^-(a_0 \tau, \tau) = 0$ along the first signal. We can integrate eq. (22a) to obtain:

$$J_1^- \Big|_{\tau}^t = ba_0^2 \tau \left[\frac{1}{2p-1} \left(\frac{t'}{\tau}\right)^{2p-1} - \frac{1}{p} \left(\frac{t'}{\tau}\right)^p \right] \Big|_{\tau}^t \quad (25)$$

After substituting the limits, we can use Eq. (24b) to write Eq. (25) in a simpler form with the definition $\tilde{a} = a_* / a_0$.

$$J_1^-(\tilde{a}, t) = ba_0^2 t \left[C_1 \tilde{a}^2 - C_2 \tilde{a} + (C_2 - C_1) \tilde{a}^{\frac{-1}{p-1}} \right] \quad (26)$$

where $C_1 = 1/(2p-1)$ and $C_2 = 1/p$.

The integral in Eq. (25) diverges for $2p-1=0$ or $\gamma=5/3$. In this case, the integral is a logarithm. It is also easy to see that choosing $p=0$ in the second term of the source term gives a value of $\gamma=3$ which is outside the physical range (i.e., kinetic theory) $1 < \gamma < 2$. For different kinds of ducts, source terms or initial conditions, similar situations may arise and must be treated individually. It may be seen that for this particular case of boundary conditions and exponential duct area, this situation will not occur in the differential equations for the Riemann invariants greater than first order. The solution for J_1^- for the special case when $\gamma=5/3$ is:

$$J_1^- = 6a_0^2 t \left[\tilde{a}^2 - \tilde{a}^2 \ln \tilde{a} - \tilde{a} \right] \quad (27)$$

B. Solution for J_1^+

Having obtained the solution for J_1^- , this can be substituted into Eq. (18b) and integrated to find J_1^+ . The partial derivative of J_1^+ is a purely a function of t and the form of Eq. (22b) immediately suggests the integrating factor $t^{k\lambda_1}$. Then we have an ODE for $\varphi = t^{k\lambda_1} J_1^+$ along C^+

$$\frac{D\varphi}{Dt} = - \left\{ u_* a_* + k\lambda_2 \frac{J_1^-(\tilde{a}, t)}{t} \right\} t^{k\lambda_1} \quad (28)$$

Along a C^+ , u_* and a_* are individually constant and it follows that so are the terms in the braces in Eq. (28), and they do not participate in the integration. Thus,

$$J_1^+(\tilde{a}, t) = -\frac{t}{2} \left(1 - \left(\frac{a_0 k}{x} \right)^2 \right) u_* a_* - \left[\frac{k\lambda_2}{2} \left(1 - \left(\frac{a_0 k}{x} \right)^2 \right) + \left(\frac{a_0 k}{x} \right)^2 \right] J_1^-(\tilde{a}, t) \quad (29)$$

Having obtained both first-order invariants, the velocity and speed of sound can be found by their appropriate linear combinations. J_1^- is not a constant in the plane, neither are the successive J_n^- and since successive terms in the expansion contribute smaller and smaller amounts – at least in domains where the series is asymptotic – we may safely conclude that the quantity $J^- = u - 2a/(\gamma - 1)$ is no longer constant in the region that was originally a simple wave prior to the introduction of the inhomogeneity^{8,9}. It is clear that J_1^\pm are both polynomial functions of \tilde{a} and t . We will later show that this is true for all further terms in the series.

C. Plots of the wavefronts

To plot flow solutions, the equations are scaled. A length scale, time scale and a velocity scale are needed for scaling the equations. The length scale $1/\varepsilon$ and a velocity scale a_0 , the speed of sound in the stationary gas are already present in the problem. If we chose the scaling relations as below, the equations retain their original form.

$$u_*' = \frac{u_*}{a_0} \quad u_n' = \frac{\varepsilon^n u_n}{a_0} \quad a_n' = \frac{\varepsilon^n a_n}{a_0} \quad x' = \varepsilon x \quad t' = \frac{t}{\varepsilon a_0} \quad (30)$$

Figure 7 shows that the first-order correction term to the velocity has compressive and expansive parts, which means that its partial derivative with respect to x is positive in one region and negative in another. From Eq. (26), we see that J_1^- has the form $f(x+ct)$, which represents a wave traveling in the negative x direction. J_1^- is purely expansive and while J_1^+ contributes the compressive part. So, we might tentatively interpret the shape of the expansion front in the following fashion: the leading edge of the wavefront is continuously reflected due the varying duct area and sends back an expansive reflection, which in turn encounters a reduction in area in the negative x direction and reflects a compressive disturbance. This seems to be indicated in the integration procedure as well since J_1^- serves as a source term in the J_1^+ equation. The superposition of these waves finally produces a compressive/expansive wave. This is as expected, since purely expansive or compressive simple waves are no longer possible in the presence of inhomogenities^{7,9}.

D. Constant velocity case

When there is a jump in the initial conditions as in the constant velocity case with velocity lower than sonic as described in section II, there are problems when finding higher order terms. If we continue to use the characteristic structure shown in Fig. 2 and defined by Eq. (8), we find that a discontinuity occurs at the C^+ characteristic separating the expansion fan from the uniform region. This does not tally with numerical results, and indeed is even intuitively unphysical. While it is possible that higher-order terms may correct the discrepancy, it is unlikely. The more promising approach would involve taking the limit of a continuous case just as we did for the homogeneous case. The limit must be taken numerically as the analytical complexity is prohibitive.

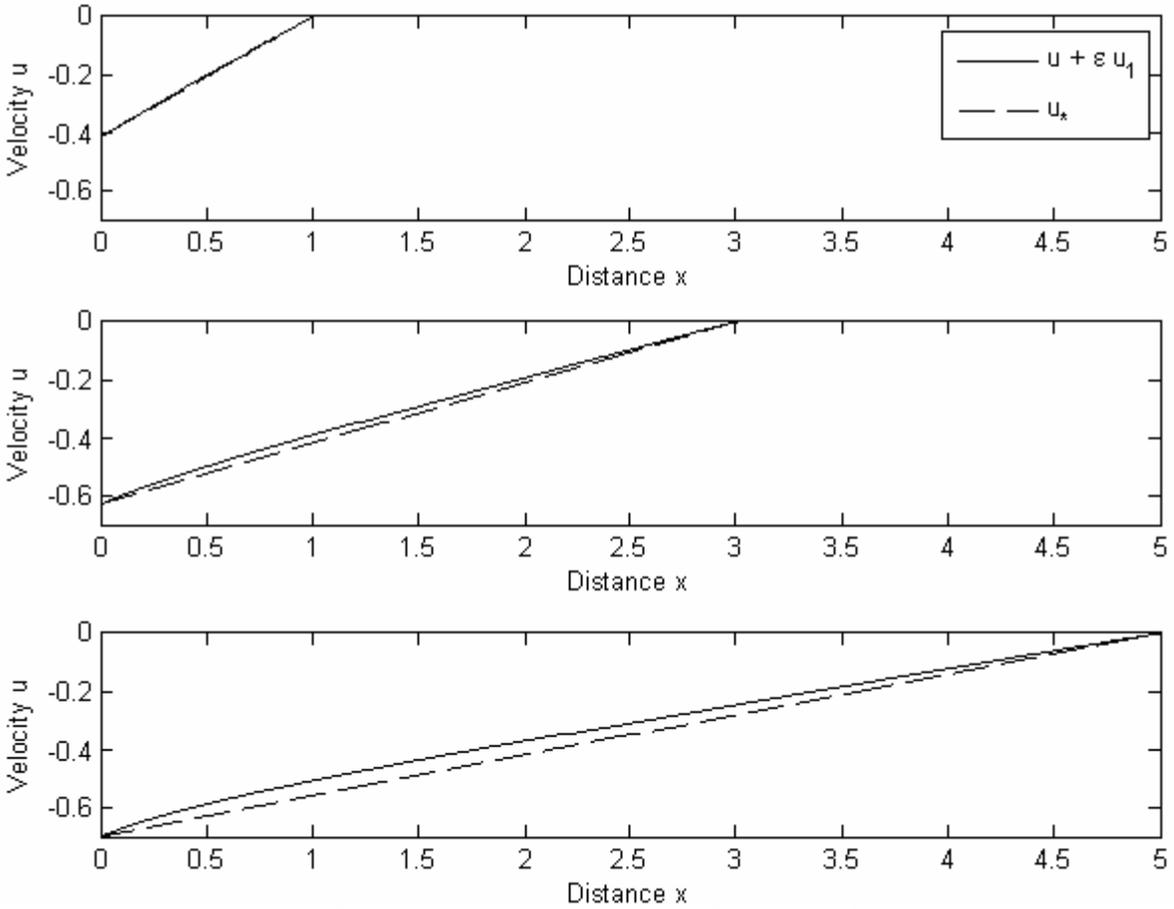


Figure 6a, b, c. Velocity profile for boundary condition specified by Eq. (9) ($k=1$) and exponential duct ($\epsilon = 0.05$) a: Time = 1; b: Time = 2; c: Time = 5.

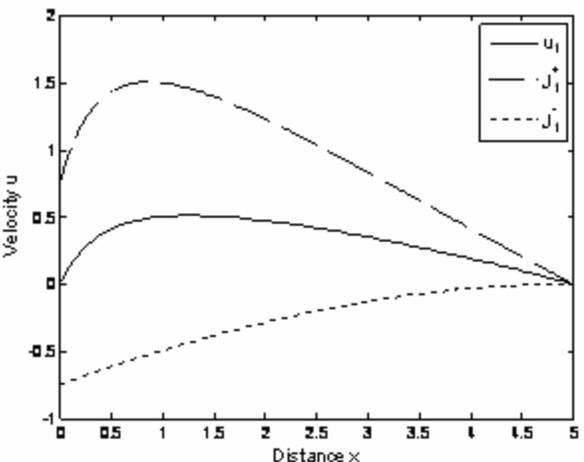


Figure 7 First order Riemann invariants.

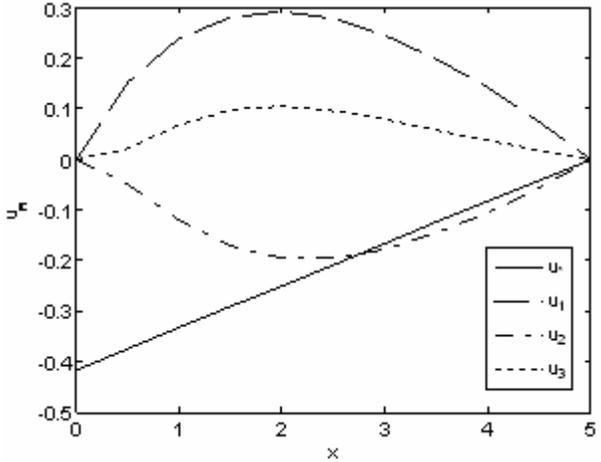


Figure 8. Numerically integrated terms up to third order.

E. Numerical Integration for higher order terms

While analytical expressions may still be obtained for each successive higher-order term, the algebraic manipulation involved is severely limiting. Hence we resort to numerical integration and indeed, this is unavoidable in a general case. In short, given any x, t we find the equations of both characteristics passing through them and evaluate the integrals in Eqs. (17) and (19) numerically. It may be seen in Fig. 9 that the higher order terms alternate in sign and are of similar orders of magnitude. Thus, the contributions of higher order terms to the series depend most strongly on ε , satisfying the asymptoticness requirement when ε is sufficiently small. It follows that addition of the j^{th} term to the asymptotic series will produce a correction of order ε^j . Since our integration algorithm was simplistic, computational time for the higher order terms increased geometrically.

F. A note on the special case of $\gamma = 5/3$

The first-order velocity term for the special case of $\gamma = 5/3$ is shown against other values of gamma in Fig. 9. This is the limiting value for a monatomic gas predicted by kinetic theory. Confirming physical intuition, the variation is continuous across the range $1 < \gamma < 2$. A similar situation does appear in many different situations in gas dynamics. For example, in the purely homogenous case, as stated before, there are in general three waves, each propagating on its family of characteristics. In a typical initial value problem, u , a and s are uniform everywhere but in a particular region $a \leq x \leq b$ at $t = 0$. The three waves propagate, interact in a particular region and then separate into three distinct simple waves. The solution in this interacting region is usually calculated numerically, but in the case of isentropic flow, we may perform a hodograph transformation to reduce the problem to solving linear ODEs. The values of $\gamma = 5/3$ and $\gamma = 7/5$ appear as special cases in the solution of this equation¹. In an analysis of the wavefront expansion technique, there is a rescaling of variables for $\gamma = 5/3$ that allows considerable simplification to be made². For our problem, however, no direct physical or analytical implication is seen immediately, although intriguing questions remain as to why this value appears.

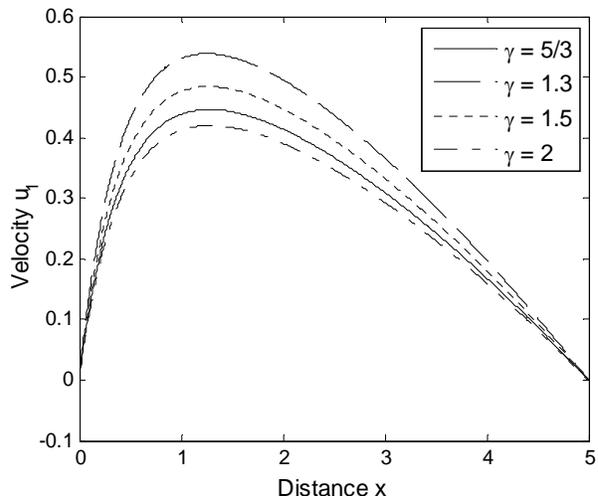


Figure 9. Effect of gamma on the first-order velocity.

G. Comparison with CLAWPACK

The first order analytical solution is compared with results from CLAWPACK, a finite-volume simulation software for hyperbolic conservation laws¹⁰. The results compare very well but with a small deviation at the boundary. This might be because of the way CLAWPACK implements boundary conditions through ‘ghost cells’. The ghost cells lie outside the domain of interest and an integration of the conserved variables must be performed. This integral is approximated to 1st order by using the value of the integrand at the middle of the ghost cell. It can be found more accurately by integrating the extrapolated known leading order solution in the ghost cell; however, this is ignored since it is more a technicality rather than an error. Finally, Figs. 10a-e show that the first order asymptotic solution is accurate even at large times of up to $10T$, where T is the time scale in the problem defined by Eq. (30).

H. Convergence and Asymptoticness

The relative magnitudes of the leading order and first order terms in the expansion are shown in Fig. 7. Numerical integration shows that at least the first few perturbation terms are of similar order of magnitude, implying that their contribution to the solution series is dictated by the value of ε . Both Fig. 7 and 8 show the velocity profiles at $5T$. Loosely, given that we know terms up to n^{th} order, the error is of order ε^{n+1} .

It is possible to make a few observations. First, for $\gamma \neq 5/3$, we make the claim that for the kind of plane wave we considered, every Riemann invariant is a polynomial in \tilde{a} multiplied by a power of t . That is,

$$J_1^\pm(\tilde{a}, t) = t^n P_n^\pm(\tilde{a}) \quad (32)$$

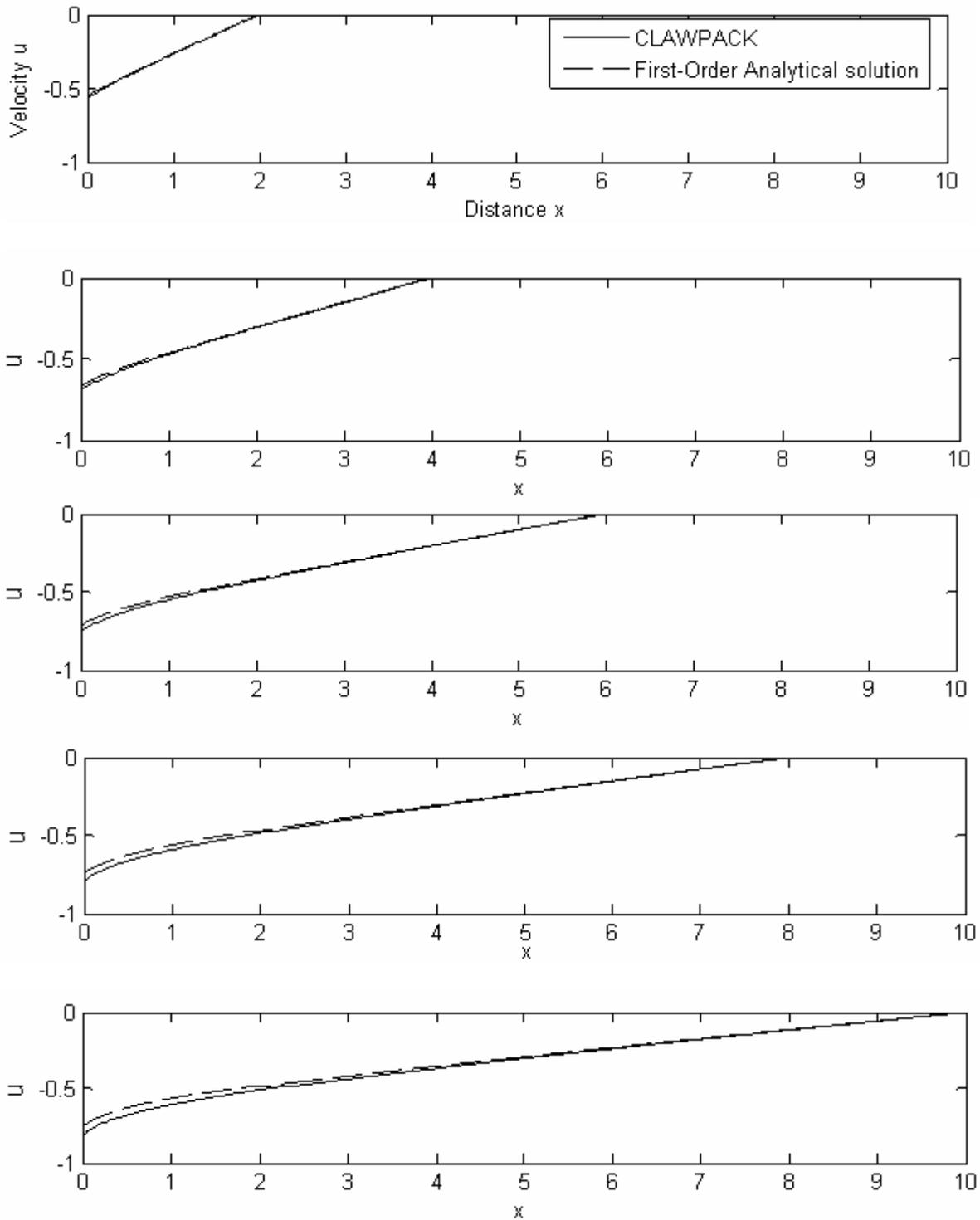


Figure 10 a-e. Comparison of CLAWPACK simulation with first order analytical solution. ($\varepsilon = 0.1, k = 1$) at times $t = 2, 4, 6, 8, 10$

where $P_n^\pm(\tilde{a})$ has terms of the form $C_k \tilde{a}^k$ with real k . The proof is by induction. We assume as our induction hypothesis that Riemann invariants order less than $n - 1$ are polynomials of the following form:

$$J_k^\pm = t^k P_k^\pm \quad \text{for } k = 1, 2, \dots, n-1 \quad (33)$$

For an exponential duct increasing in area with x , Eq. (17) becomes:

$$\frac{D^-}{Dt} J_n^- = \sum_{k=1}^{n-1} (\lambda_1 J_k^+ + \lambda_2 J_k^-) \frac{\partial}{\partial x} J_{n-k}^- + \sum_{k=0}^{n-1} u_k a_{n-1-k} \quad (34)$$

Then, note that,

$$\frac{\partial}{\partial x} \tilde{a}^i = i \tilde{a}^{i-1} \left(\frac{\gamma-1}{\gamma+1} \right) \frac{1}{t} \quad \text{for some } i \neq 0$$

Therefore, we may write Eq. (33) as,

$$\frac{D^-}{Dt} J_n^- = t^{n-1} P(\tilde{a})$$

Again, integrating a general term in the polynomial along the J_*^- characteristic,

$$\int_{\tau}^t t'^{n-1} \tilde{a}^i dt' = \int_{\tau}^t t'^{n-1} \left(\frac{t'}{\tau} \right)^{i(p-1)} dt' = \frac{1}{i(p-1)+n} t^n \left(\tilde{a}^i - \tilde{a}^{i \frac{-n}{p-1}} \right)$$

Hence, we may write J_n^- in the form given by Eq. (34). The proof for J_n^+ is similar. But we have shown earlier by direct integration that the first order Riemann invariants are in the prescribed form. Therefore, by induction all terms in the asymptotic series are of the same form. Then, the asymptotic series for the velocity and speed of sound is just:

$$\begin{aligned} u &= u_* + a_0 \left\{ (a_0 \varepsilon t) P_1 + (a_0 \varepsilon t)^2 P_2 + \dots + (a_0 \varepsilon t)^n P_n \dots \right\} \\ a &= a_* + a_0 \left\{ (a_0 \varepsilon t) P_1 + (a_0 \varepsilon t)^2 P_2 + \dots + (a_0 \varepsilon t)^n P_n \dots \right\} \end{aligned}$$

The polynomials $P_i (i=1, 2, \dots)$ are continuous functions of \tilde{a} $\tilde{a} \in [a - (\gamma-1) \max\{u(0, t)\} / 2, 1]$ and hence have compact support. Then, it follows by the Weierstrass theorem that they must have maxima $P(\tilde{a}_i^0) < \infty$ for each i . Now the question remains, is $C = \sup_i [P_i(\tilde{a}_i^0)] < \infty$.

From the initial few terms obtained by numerical integration (Fig. 8), it certainly seems like it does, and if so, the asymptotic series is dominated by the geometric sequence $C \sum (a_0 \varepsilon t)^n$ that is convergent for $a_0 \varepsilon t < 1$ and hence converges uniformly (at a fixed t) to the actual solution of the problem. This very intuitive; the condition just states that the asymptotic series converges for $t < T$, with T the timescale defined by Eq. (30). Even if the series does not converge, the first few terms of the asymptotic expansion provides a good approximation as we have already seen.

VI. Conclusion

This study provides a flow solution to the Euler equations for a particular kind of inhomogeneity in the form of an asymptotic expansion. We present a method to integrate and find each term of the series for simple wave regions: those that have only one family of characteristics propagating through it. The method is then illustrated for a model problem where simple closed-form expressions for each term could be obtained. Nonlinear distortion of the

wavefront is present in the analytical solution and confirmed by numerical simulation. We also see what may be interpreted as a reflection.

We solved the equations for the particular case of an exponential duct, but they may be numerically integrated for a general area variation and general initial conditions. Problems with spherical or cylindrical symmetry may be solved as well since they have source terms similar to those introduced by area variation when the symmetry is exploited to reduce the dimensionality of the problem. For isentropic flow, the equations have the form:

$$\frac{\partial}{\partial t} J^{\pm} + (u \pm a) \frac{\partial}{\partial r} J^{\pm} = \mp j \frac{ua}{r} \quad j = 1, 2 \text{ for cylindrical, spherical waves}$$

The solution to this problem is analogous to the one solved in the paper. There will, in general, be a shock at the origin on account of the radial distance going to zero and that must be handled carefully.

This method may find application in numerical solutions of the Euler equations where source terms have this particular form. For example, CLAWPACK uses a finite-volume method that involves solving Riemann problems in each cell (Godunov-type method)¹⁰. When source terms are present, they are assumed to be usually functions of the conserved quantity q and not on any of derivatives. This is true in most situations where source terms arise due to entropy/density gradients or geometrical constraints. Now, at each cell location an ODE has to be solved:

$$q_t = \psi(q, x)$$

where ψ is a source term and q is a conserved quantity like density or momentum. This ODE is generally solved by a Runge-Kutta type method. We might, instead, use the asymptotic expansion to solve the appropriate Riemann problem in each grid-cell by integrating the terms numerically to obtain higher orders of accuracy at conceivably lower computational cost.

The technique of introducing another scale into the problem is sometimes called the method of *multiple-scales* and it lends itself to other inhomogeneous situations like, say, the presence of an initial entropy gradient in the duct. This entropy gradient may be assumed to be characterized by a length scale and then we can construct asymptotic expansions for flow velocity, speed of sound and the entropy as well. Clearly, we must solve three coupled equations in this case, but under special circumstances, one (or two) equations may be solved independently and substituted into the other just as was done in this paper.

The solution is valid only in a simple wave region of the t - x plane and interacting regions must be examined. For completeness, it must be extended to handle discontinuities in the flow: shocks. Generally, a ‘weak solution’ in the integral equation sense is formulated, where discontinuous solutions are used to represent shocks. Although the general problem is difficult, when only weak shocks are present in simple wave regions, the entropy change across the shock is small and approximate solutions can be constructed more easily^{1,7-9}. A similar procedure can be attempted when source terms are present.

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