Lec 19 (Finishing up Alexander's Theorem)

Back to FPP

Theorem If

(#4) 1) $F(0) < p_c$ $\quad$ $\exists$ a $C > 0$ st if $|x| \geq C$

2) $\mathbb{E}[e^{\alpha \tau_e}] < \infty$ $\quad$ then for sufficiently large n

$\exists$ a lattice path 0 to nx with "a Q_x skeleton" of

at most $m \leq 2n+1$ vertices.

Main ingredient is Kesten's bound: If conditions in (#4) hold then

$$P\left(\left| T(0,x) - \mathbb{E}T(0,x) \right| \geq u|x|^{1/2} \right) \leq C_1 e^{-C_2 u}$$

for $u < C_3 |x|$

☆ Notice that although we have proved Gaussian concentration with more modern techniques, Alexander only uses Kestens exponential concentration.

Define $S_x(y) = \mathbb{E}T(0,y) - g_x(y) = h(y) - g_x(y)$

$\geq g(y) - g_x(y) \geq 0$ (by previous)

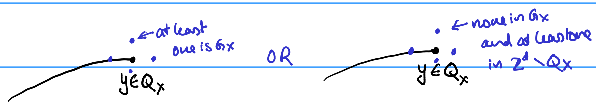
$S_x(y+z) \leq S_x(y) + S_x(z)$ (Using subadditivity of h and

$Q_x(1/2, \varphi)$ $\quad \varphi = \log t$

$G_x = \{ y \in \mathbb{Z}^d : g_x(y) > g(x) \}$

$\Delta_x = \{ y \in Q_x : y$ is adjacent to $\mathbb{Z}^d \setminus Q_x$, y not adjacent to $G_x \}$

$D_x = \{ y \in Q_x : y$ is adjacent to $G_x \}$.



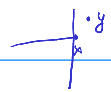
We can always assume at least one neighbor will be in $\mathbb{Z}^d \setminus Q_x$ when y is in the Q_x skeleton ($y = v_i$) since v_i is always chosen to be maximal (next site is not in Q_x)

Lemma:

1) If $y \in Q_x$ then $g(y) \leq 2g(x)$ and $|y| \leq 2d|x|$

2) If $y \in \Delta_x$ then $S_y(y) \geq c|x|^{1/2} \log|x|$ ← like a v_i but have not "gone past" $g_x(x)$

3) $y \in D_x$, then $g_x(y) \geq \frac{5g(x)}{6}$ ← like a $v_i \in Q_x$ but have gone past $g_x(x)$

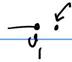


There are reasonable things to believe of course.

1) If $y \in Q_x$ is GOOD, $g_x(y) \le g(x)$ and so on, so

$g(y) \le 2g(x)$ (Can't be too bad)

$|y| \le 2d|x|$ (y can't be too far away otherwise $g_x(y) > g(x)$)

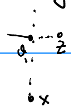
2) Basically $\exists$ a  $z \notin Q_x$ so $S_z(z) \geq C|x|^{1/2} \log|x|$

(Is bad). So $S_x(y) \geq S_x(z) - S_x(e_k)$

this can be bounded.

USES Maximality of v_i

3) This is fairly straightforward as well.

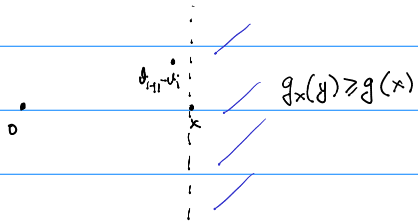
 $\leftarrow g_x(z) > g(x)$ So $g_x(v_i)$ had better be reasonably big too.

Using this lemma, we can define long and short increments.

$S((v_i)) = \{i : 0 \le i < m-1,\ v_{i+1} - v_i \in \Delta_x\}$ (SHORT)

$L((v_i)) = \{i : 0 \le i < m-1,\ v_{i+1} - v_i \in D_x\}$ (LONG)

The long reason is obvious:



<u>Prop 3.4</u>: If $|x| \geq C$, then for large enough n

$\exists$ a lattice path γ_x from 0 to nx with a Q_x skeleton

of at most $m \leq 2n+1$ vertices.

Idea: There is going to be a lot of points on γ in Q_x

(by BK inequality).

There cannot be too many SHORT increments since

$S_x(y) > C\,|x|^{1/2} \log|x|$ and this would violate GAP

There can not be too many long increments for a

similar reason. This bounds m.

Pf: $\quad Y_i = ET(v_i, v_{i+1}) - T(v_i, v_{i+1})$

$$0 \leq Y_i \leq ET(v_i, v_{i+1}) \leq E[\tau_e]\,|v_{i+1} - v_i|$$

$$\Rightarrow \quad P(Y_i \geq u|x|^{1/2}) = 0$$

where $\quad u \approx C\,|v_{i+1} - v_i|$

But by previous lemma (i) $|v_{i+1} - v_i| \leq 2d|x|$ — (#3)

So for $\quad 0 \leq u \leq C\,|v_{i+1} - v_i|$

$$P(Y_i \geq u|x|^{1/2}) \leq P\left(Y_i \geq u \frac{|v_{i+1} - v_i|^{1/2}}{\sqrt{2d}}\right)$$

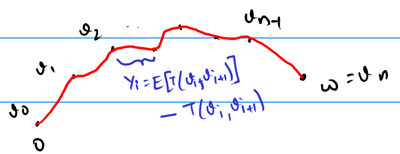
$$\leq C e^{-cu}$$

— (#4) Follows from the concentration inequality.

$$\Rightarrow E\left[e^{\beta_0 Y/|x|^{1/2}}\right] \leq \frac{C}{C - \beta_0}$$

(exponential bound)

Let $T(0, w; (v_j))$ be the minimum passage time among all lattice paths from 0 to w with Q_x skeleton (v_j).



Its the minimum over all possible red paths.

Let $\{Y_i'\}_{i=1}^{n}$ be independent copies of the Y_i

that are also independent of each other.

By the BK inequality

$$P\left(\sum_{i=0}^{m-1} Y_i \geq t\right) \leq P\left(\sum_{i=0}^{m-1} Y_i' \geq t\right)$$

One has to be careful here, and ought to

perhaps use Kesten's form of the inequality.

$$P\left(\bigcup A_{m(i,j)} \circ \cdots \circ A_{m(i,k(i))}\right)$$

$$\leq \sum_j \prod P\left(A_{m(i,j)}\right)$$

$$\Rightarrow P\left(\sum_{i=0}^{m-1} ET(\vartheta_i, \vartheta_{i+1}) - T(0, \vartheta_m; (\vartheta_j))\right.$$

$$\left. > Cm|x|^{1/2}\log|x|\right)$$

$$\leq P\left(e^{\beta \sum_{i=0}^{m-1} Y_i'/|x|^{1/2}} > e^{\beta Cm \log|x|}\right)$$

$$\leq e^{-\beta Cm \log|x|} \, \mathbb{E}\left[e^{\beta \sum_{i=0}^{m-1} Y_i'/|x|^{1/2}}\right] \qquad \text{—(\#3)}$$

POLL: Have you have seen this last inequality?

YES NO

(#5) becomes

$$\le e^{-\beta C_1 m} C_2^m = e^{-C_3 m} \quad \text{—(\#6)}$$

You can adjust constants so that this is as small as necessary.

We know $|Q_x| \le 2^d |x|^d$ (by lemma 3.3 (i))

$\Rightarrow$ (#6) says that

The # of Q_x skeletons is $(2^d |x|^d)^m$

Thus $P\left(\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T(0, v_m; v_j) > C\, m |x|^{1/2} \log|x| \right.$ for some Q_x skeleton

$$\le (2^d |x|^d)^m e^{-C(\beta) m} \quad (\text{from \#6}) \quad \text{—(\#7)}$$

The KEY is that we can adjust β and the constants as necessary to make (#7) decay exponentially in m.

Sum (#7) over m to get

$$P\Big(\sum_{i=0}^{m-1} ET(v_i, v_{i+1}) - T(0, v_m; v_j^\circ) > C\, m |x|^{1/2} \log|x| \Big)$$

for <u>some</u> Q_x skeleton and for some m

$$\leq 2e^{-(C \log|x|)} \qquad -(\#8)$$

How large can m be? How small can it be?

If you have an m Q_x skeleton

$$\underset{\text{increments in } Q_x}{m g(x) \geq} \sum_{i=0}^{m-1} g_x(v_{i+1} - v_i) = g_x(nx)$$

$$= n g(x)$$

$P(T(0, nx) \leq n g(x) + n) \longrightarrow 1$ (since $\frac{T(0,nx)}{n} \to g(x)$ a.e)

$-(\#8a)$

Combining (#8) and (#8a) we choose n large enough st the intersection of the events in (#8) and (#8a) have large probability.

for some fixed skeleton?

$T(0, nx, (\vartheta_j)) = T(0, nx) \le ng(x) + n$

for some Q_x skeleton.

Basically what you have to do is to fix a path γ in the environment and find the sites $(\vartheta_1, \cdots, \vartheta_m)$ for the particular geodesic γ.

Then by #8

$$\sum_{i=0}^{m-1} \mathbb{E}\, T(\vartheta_i, \vartheta_{i+1}) - T(0, nx, (\vartheta_j))$$

$$\le C m |x|^{1/2} \log|x|$$

$$\le ng(x) + n + Cm|x|^{1/2} \log x \le \underbrace{ng(x) + 2Cm|x|^{1/2}\log|x|}_{(\#9)}$$

using $m \ge n$.

$$\sum_{i=0}^{m-1} \mathbb{E}\, T(\vartheta_i, \vartheta_{i+1}) = \sum_{i=0}^{m-1} \Big(g_x(\vartheta_{i+1} - \vartheta_i) + \underbrace{s_x(\vartheta_{i+1} - \vartheta_i)}_{\mathbb{E}\, T(\vartheta_i, \vartheta_{i+1}) - g_x(\vartheta_{i+1} - \vartheta_i)} \Big)$$

$$\ge \underbrace{g_x(nx)}_{\text{linearity}} + \underbrace{|S(\vartheta_i)|}_{\#\text{ of shad increments}}\, C|x|^{1/2} \log|x|$$

Combining with (#9) we get

$$g_x(nx) + |S(\vartheta_i)|\, C_1 |x|^{1/2} \log|x| \le ng(x) + 2C_2 m|x|^{1/2} \log|x|$$

I think $g_x(nx) = g\left(\left(nx \cdot \frac{x}{|x|}\right) \frac{x}{|x|}\right)$

$$= g(nx)$$

$\Rightarrow$ The # short increments is bounded.

$$|S(\vartheta_i)| \leq \frac{C_2 m}{C_1} \leq \frac{m}{4}.$$

Again, you need to be careful with the constants here!

Next to bound long increments

$$ng(x) + 2Cm\sqrt{|x|}\log|x| \geqslant \sum_{i=0}^{m-1} \mathbb{E}T(\vartheta_i, \vartheta_{i+1})$$

$$= \sum_{i=0}^{m-1}\Big(g_x(\vartheta_{i+1} - \vartheta_i) + \underbrace{s_x(\vartheta_{i+1} - \vartheta_i)}_{\geqslant 0}\Big)$$

$\geqslant |L(\vartheta_i)| \dfrac{5g(x)}{6}$ using the definition of long increments.

$$|L(v_i)| \leq \frac{6n}{5} + \frac{2Cm\sqrt{|x|}\,\log|x|}{g(x)}\,\frac{6}{5}$$

Then we $g(x) \geq C|x|$ since its a norm.

$$\leq \frac{6n}{5} + \frac{m}{8}$$ (by choosing $|x|$ large but finite)

So

$$m = |S(v_i)| + |L(v_i)| \leq \frac{6n}{5} + \frac{m}{8} + \frac{m}{4}$$

$$\Rightarrow \quad m\frac{5}{8} \leq \frac{6n}{5} \quad \Rightarrow \quad m \leq 2n$$

There is an interplay in the last part going from the random v_i to the deterministic that I haven't completely digested. But this will have to do, in the interest of time.