The KKL Inequality ~ 1988

Influence + Bool Fncs:

$\{0,1\}$. $\{-1,1\}$. ~ Easier to work with.

Def: Boolean Functions: $f : \Omega_n \to \{0,1\}$, $\Omega_n = \{-1,1\}^n$

Def: k pivotal for f for ω if $f(\omega) \neq f(\omega^i)$ ~ ω^i has ith bit flipped. $k \in [n]$

Def: Influence – $I_k(f) := P(k \text{ pivotal for } f)$

Ex: $f(x_1, \ldots, x_n) = x_1 \to I_1(f) = 1,\ I_k(f) = 0,\ k \neq 1.$

$\rightsquigarrow \frac{1}{n} \sum I_k(f) = \frac{1}{n}.$

Bounds on Influence:

Discrete Poincaré: $f : \Omega_n \to \{-1,1\}$ then $\mathrm{Var}(f) \leq \sum I_i(f)$,

so $I_i(f) \geq \mathrm{Var}(f)/n$, some i. ~ Efron-Stein

... KKL gives logarithmic improvement ⇝

KKL: $\exists c > 0$ s.t. if $f : \Omega_n \to \{0,1\}$, then $\exists i$ s.t.

$$I_i(f) \geq c \,\mathrm{Var}(f) \log n / n.$$

Ex: Bound is sharp ⇒ Ben-Or + Linial give an example

--- Part $[n]$ into blocks of length $\log_2(n) - \log_2(\log_2(n))$. Set f_n to be 1 if $\exists$ block containing all 1s and 0 ow. $1, \ldots, n$.

(Variance away from 0 + influences smaller than $c(\log n)/n$, $c < \infty$)

$$I_i(f_n) \leq c \,\mathrm{Var}(f_n) \log n / n. \quad \forall n \geq 1.$$

A Bit of Fourier analysis on Ω_n:

$(\Omega_n, 2^{\Omega_n}, \mathbb{P})$, $\mathbb{P} = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1)^{\otimes n}$ ~ Uniform meas.

Give $f, g \in L^2(\Omega_n)$ the inner product:

$$\langle f, g \rangle := \mathbb{E}[fg]$$

Def: $S \subseteq [n]$, $\chi_S(x) := \prod_{i \in S} x_i$. ~ Forms ortho basis for $L^2(\Omega_n)$ ~ characters of Ω_n as a group.

We view Ω_n as the group $\mathbb{Z}_2^n$. $S = \{1, 3, 4\} \subseteq [4]$, $\to x = (1011)$.

$$\chi_S(xy) = \prod_{i \in S} x_i y_i = \prod_{i \in S} x_i \prod_{i \in S} y_i = \chi_S(x)\chi_S(y).$$

$\Rightarrow$ $\boxed{f = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S}$, $\hat{f}(S) = \mathbb{E}[f\chi_S]$. (Walsh coeffs)

How does this tangent connect to the original question of 'influence'?

Prop: $f : \Omega_n \to \{0,1\}$ then:

$$I_k(f) = 4 \sum_{S: k \in S} \hat{f}(S)^2 \quad \text{and}$$

sum of influences

"in direction of k^{th} bit"

Proof: Introduce the discrete derivative: flip k^{th} bit

$$\nabla_k f : \Omega_n \to \{-1, 0, 1\} \qquad \nabla_k f(\omega) = f(\omega) - f(\omega^k)$$

Easy to see that $\nabla_k f(\omega) = \sum_{S \subseteq [n]} \hat{f}(S)\left[\chi_S(\omega) - \chi_S(\omega^k)\right] = \sum_{S: k \in S} 2\hat{f}(S)\chi_S(\omega)$

$$\Rightarrow \widehat{\nabla_k f}(S) = \begin{cases} 2\hat{f}(S) & k \in S \\ 0 & \text{o/w} \end{cases}$$

$I_k(f) = \|\nabla_k f\|_1$ is clear. Also $\|\nabla_k f\|_1 = \|\nabla_k f\|_2^2$ ✓

$\Rightarrow$ proposition by Parseval.

$$\Rightarrow \|\nabla_k f\|_2^2 = \sum_S \widehat{\nabla_k f}(S)^2 = \sum_{S: k \in S} (2\hat{f}(S))^2$$

$$\nabla_k f(\omega) = f(\omega) - f(\omega^k) = \sum_S \hat{f}(S)\chi_S(\omega) - \sum_S \hat{f}(S)\chi_S(\omega^k)$$

The idea of the proof of KKL $4\sum \hat{f}(S)^2$.

The idea of the proof of KKL $\underline{4\sum_S |S|\hat{f}(S)^2}$.

Say the influences are all small enough (o/w obviously done)

Then since $\underline{I_k(f) = \|\nabla_k f\|_2^2}$, $\underline{\nabla_k f}$ has small support.

$\Rightarrow$ $\underline{\widehat{\nabla_k f}}$ is more spread out (concentration on high freqs).

Let's try something... Take $1 \ll M \ll n$:

$$\|\nabla_k f(S)\|_2^2 \quad \underline{I(f)} = 4\sum_S |S| \hat{f}(S)^2$$

$$\sum_{0<|S|<M} \hat{f}(S)^2 \leq 4\sum_{0<|S|<M} |S| \hat{f}(S)^2$$

$$\sum_k \sum_{0<|S|<M} \widehat{\nabla_k f}(S)^2 = \sum_{0<|S|<M} \sum_k \widehat{\nabla_k f}(S)^2$$

$$\underline{\mathrm{Var} f} = \sum_k \sum_{0<|S|<M} \widehat{\nabla_k f}(S)^2 = \sum_{0<|S|<M} |S| (2\hat{f}(S))^2.$$

$$\underline{\ll} \sum_k \|\widehat{\nabla_k f}\|_2^2 \quad \left(\begin{array}{l}\text{Uncertainty principle} \\ \text{... support on higher freqs.}\end{array}\right)$$

$$= \underline{I(f)} \quad (\text{Parseval})$$

$$\chi_\phi \mapsto 1$$

$$\hat{f}(\phi)^2 = \mathbb{E}[f]$$

We have $\sum_{|S|>0} \hat{f}(S)^2 = \underline{\mathrm{Var}(f)}$.

If $\underline{\text{frac}}$ of Fourier mass of f conc. below M,
$I(f)$ $\underline{\text{bigger}}$ than $\underline{\mathrm{Var}(f)}$, so $\underline{\text{one influence}}$ is big.

If most of $\underline{\text{mass}}$ on larger freqs, $I(f)$ $\underline{\text{large}}$ using:

$$\underline{I(f) = 4\sum_S |S| \hat{f}(S)^2}.$$

... How do we understand the spectral info?

One More Tool we need before the proof of KKL:

Hypercontractivity: start w/ a rough function.... wait to smooth it out:

Similar to heat flow. $\|Tf\|_p \le \|f\|_p$.

For $f \in L^q(\mathbb{R}^n)$, $\|K_t * f\|_2 \le \|f\|_q$, $1<q<2$.

K_t is the heat kernel: $K_t(x) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right)$

with Gaussian meas. $\gamma^n(A) = (2\pi)^{-n/2} \int_A \exp(-\frac{1}{2}|x|^2) \, d\lambda^n(x)$.

Now in the case of the hypercube:

Def: $\omega \in \Omega_n$. Then ω_ρ is the distr. of x s.t. $x_i = \omega_i$ w/ prob ρ and x_i is rerandomized w/ prob $1-\rho$.

Def: $f: \Omega_n \to \mathbb{R}$, $T_\rho f(\omega) = \mathbb{E}_{x \sim_\rho \omega}[f(x)]$. $\rho \in [0,1]$, $\omega \in \Omega_n$.

Ex: $T_0 f = \mathbb{E}[f]$. $T_1 f = f$.

Easy to see T_ρ

$$T_\rho \chi_S(\omega) = T_\rho\left(\prod_{i\in S} \omega_i\right) = \prod_{i\in S} \mathbb{E}_{x \sim_\rho \omega}[\omega_i] = \prod_{i \in S} \rho\omega_i + \frac{1}{2}\underbrace{(-1+1)}_{0}$$

$$\rho^{|S|} \prod_{i\in S} \omega_i = \rho^{|S|}\chi_S(\omega).$$

Theorem – [Bonami-Gross-Beckner]: $f: \Omega_n \to \mathbb{R}$, $\rho \in [0,1]$.

$$\|T_\rho f\|_2 \le \|f\|_{1+\rho^2}.$$

- "Hypercontractivity" inequality since $\|T_\rho f\|_2 \le \|f\|_2$ a contraction, but we even have $\|T_\rho f\|_2 \le \|f\|_{1+\rho^2}$, $1+\rho^2 \le 2$.

Now we see its usefulness in the proof:

$n^{-3/4} \, \forall_{n} f \geq c \operatorname{Var}(f) \frac{\log n}{n}, \; c > 0.$

Case 1: $\exists k \in [n]$ st $I_k(f) \geq n^{-3/4} \operatorname{Var}(f)$. So clearly $I_k(f) \geq c \operatorname{Var}(f) \frac{\log n}{n}$.

Case 2: ("Small" influences):

$$\forall k, \quad I_k(f) = \|\nabla_k f\|_2^2 \leq \operatorname{Var}(f) n^{-3/4}.$$

↳ Then spectrum mostly supported on high freqs:

Take $M \geq 1$:

$$\sum_{1 \leq |S| \leq M} \hat{f}(S)^2 \leq \sum_{1 \leq |S| \leq M} |S| \hat{f}(S)^2 \leq 2^{2M} \sum_{1 \leq |S| \leq M} (1/2)^{2|S|} |S| \hat{f}(S)^2$$

$$\leq 2^{2M} \sum_{|S| \geq 1} (1/2)^{2|S|} |S| \hat{f}(S)^2$$

⤳ Chosen noise level $\rho = 1/2$. others were possible.

$f = \sum_S \hat{f}(S) \chi_S \mapsto \sum_S \rho^{|S|} \hat{f}(S) \chi_S$.

$$= \frac{1}{4} 2^{2M} \sum_k \|T_{1/2}(\nabla_k f)\|_2^2$$

$$\leq \frac{1}{4} 2^{2M} \sum_k \|\nabla_k f\|_{5/4}^2 \quad \text{(Hypercontractivity)}$$

$$\leq 2^{2M} \sum_k I_k(f)^{8/5} \quad \rightsquigarrow \|\nabla_k f\|_2^2 = I_k(f).$$

$$\leq 2^{2M} n \operatorname{Var}(f)^{8/5} n^{-\frac{3}{4} \cdot \frac{8}{5}} \quad \left(\text{Using } I_k(f) \leq \operatorname{Var}(f) n^{-3/4}\right).$$

$$\leq 2^{2M} n^{-1/5} \operatorname{Var}(f) \quad \left(\text{Since clearly } \operatorname{Var}(f)^{8/5} \leq \operatorname{Var}(f)\right)$$

⤳ $\operatorname{Var}(f) \leq 1$ as f bool.

Set $M = \lfloor \frac{1}{20} \log_2 n \rfloor$:

$$\sum_{1 \leq |S| \leq \frac{1}{20} \log_2 n} \hat{f}(S)^2 \leq n^{1/10 - 1/5} \operatorname{Var}(f) = n^{-1/10} \operatorname{Var}(f).$$

... So most of spectrum conc above $\Omega(\log_2 n)$.

Thus we get:
$$\sup_k I_k(f) \geq \frac{\sum_k I_k(f)}{n} = \frac{4\sum_{|S|\geq 1} |S|\hat{f}(S)^2}{n}$$
$$\geq \frac{1}{n}\left[\sum_{|S|\geq M} |S|\hat{f}(S)^2\right]$$
$$\geq \frac{M}{n}\left[\sum_{|S|\geq M} \hat{f}(S)^2\right]$$
$$= \frac{M}{n}\left[\mathrm{Var}(f) - \sum_{1\leq |S|\leq M} \hat{f}(S)^2\right]$$
$$\geq \frac{M}{n}\mathrm{Var}(f)\left[1-n^{-1/10}\right]$$
$$\geq c_1 \mathrm{Var}(f)\frac{\log n}{n},$$
where $c_1 = \frac{1}{20\log 2}(1-2^{-1/10})$. $\square$

Applications of the KKL Inequality:

KKL gives a cleaner proof of the celebrated Harris-Kesten result ($p_c = 1/2$ on $\mathbb{Z}^2$ w/ Bernoulli weights)

Ref: https://arxiv.org/pdf/math/0410359.pdf

Why do computer scientists care? – Implications to crypto.

– First time harmonic analysis has been applied to this kind of result.

Nice explanation of these connections and more: http://www.math.chalmers.se/~steif/homepage.pdf

Applications to social choice:

View $f: \Omega_n \to \{0,1\}$ as a "voting function"

That is, given the binary choice of n citizens, make some decision.

Ex Dictatorship $f(x_1, \dots, x_n) = x_1$ (Only person 1 matters)

- Nice Reference for connections to social choice:

https://math.mit.edu/~elmos/fall19/survey.pdf

Another example: $f: \{-1,1\}^n \to \{0,1\}$ monotone "voting" function,

KKL $\Rightarrow$ you can "bribe" $O\left(\frac{1}{\log n}\right)$ fraction of voters to swing the vote in favor of some $b \in \{0,1\}$.

Note:

In the original KKL paper, a corollary is given that states there is a set of $\frac{n}{\log n} \omega(n) = o(n)$ variables which dominates f (By repeated application of KKL).