Semigroup [Chatterjee]

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Let $(X_t)_{t\geq 0}$ be a Markov process, a semigroup operator $(P_t)_{t\geq 0}$ defines as

$$\underline{P_t(f(x)) = \mathbb{E}(f(X_t) \mid X_0 = x)}$$
$$= \mathbb{E}_x(f(X_t))$$

$$P_0 = Id, \quad P_{t+s} = P_t P_s = P_s P_t$$

$$P_{t+s}(f(x)) = \underline{\mathbb{E}_x(f(X_{t+s}))}$$
$$= \mathbb{E}_x(\underbrace{\mathbb{E}_x(f(X_{t+s}) \mid \mathcal{F}_s)})$$
$$= \mathbb{E}_x(\mathbb{E}_{X_s}(f(X_{t+s})))$$
$$= \mathbb{E}_x(P_s(f(X_t)))$$
$$= P_t P_s(f(x))$$

The generator $\mathcal{L}$ of the semigroup is defined by defined by

$$\mathcal{L}f := \lim_{t\to 0} \frac{P_t f - P_0 f}{t} = \lim_{t\to 0} \frac{P_t f - f}{t}$$

Note that

$$\partial_t P_t = \lim_{h\to 0} \frac{P_{t+h} - P_t}{h}$$

(P_t) is a semigroup

$$= \lim_{h\to 0} \frac{P_h - Id}{h} P_t$$

$$= \mathcal{L}P_t = P_t\mathcal{L}$$

$$\Rightarrow \quad P_t = \exp(t\mathcal{L})$$

Define $f, g \in \mathcal{L}^2(\mu)$, where μ is an equilibrium measure, i.e.

$$\int P_t f \, d\mu = \int f \, d\mu$$

or

$$\lim_{t\to\infty} P_t f \overset{\mathcal{L}^2}{=} \mathbb{E}_\mu(f)$$

Also, $\langle f, g\rangle_{\mathcal{L}^2(\mu)} := \int fg \, d\mu$

Dirchelet form

$$\boxed{\mathcal{E}(f,g) := -\langle f, \mathcal{L}g\rangle}$$

Lemma 1 (Covariance Lemma)

$$\mathrm{Cov}_\mu(f,g) = \int_0^\infty \mathcal{E}(f, P_t g)\, dt$$

Pf: $\mathrm{Cov}_\mu(f,g) = \mathbb{E}_\mu(f\cdot g) - \mathbb{E}_\mu f\, \mathbb{E}_\mu g$

$$= \mathbb{E}_\mu(f\cdot P_0 g) - \mathbb{E}_\mu f \lim_{t\to\infty} P_t g$$

$$= \mathbb{E}_\mu\left(f\cdot \left(P_0 g - \lim_{t\to\infty} P_t g\right)\right)$$

proper assumption for P_t

$$= -\mathbb{E}_\mu\left(f\cdot \int_0^\infty \partial_t P_t g \, dt\right)$$

$$= -\mathbb{E}_\mu\left(f\cdot \int_0^\infty \mathcal{L} P_t g \, dt\right)$$

$$= \int_0^\infty \mathbb{E}_\mu\left(f\cdot -\mathcal{L} P_t g\right) dt$$

$$= \int_0^\infty \mathcal{E}(f, P_t g)\, dt \qquad \square$$

Ornstein – Uhlenbeck Semigroup

The Standard OU process is $(X_t)_{t\ge 0}$ that satisfies SDE:

SBM

$$dX_t = -X_t\,dt + \sqrt{2}\,dB_t$$

$$\Rightarrow \quad e^t dX_t = -e^t X_t\,dt + \sqrt{2}\,e^t dB_t$$

By Itô's formula $f(t, B_t) \in C^2$

$$df = \frac{\partial f}{\partial B_t} dB_t + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial B_t^2} dt$$

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Isometry

$f \in L^2(\mu)$

$$\star \quad \mathbb{E}\left(\int f \, dB_t\right)^2 = \mathbb{E} \int f^2 \, dt$$

$$(dB_t)^2 = dt$$

$$d(e^t X_t) = \frac{\partial (e^t X_t)}{\partial X_t} dX_t + \frac{\partial (e^t X_t)}{\partial t} dt + \frac{1}{2} \frac{\partial^2 (e^t X_t)}{\partial X_t^2} (dX_t)^2$$

$$= e^t dX_t + e^t X_t dt + 0$$

$$= -e^t X_t dt + \sqrt{2} e^t dB_t + e^t X_t dt$$

$$= \sqrt{2} e^t dB_t$$

$$\Rightarrow e^t X_t - X_0 = \sqrt{2} \int_0^t e^s dB_s$$

$$\boxed{X_t = e^{-t} X_0 + \sqrt{2} e^{-t} \int_0^t e^s dB_s}$$

where $\sqrt{2} e^{-t} \int_0^t e^s dB_s \sim N\left(0, \sqrt{1 - e^{-2t}}\right)$

(by using Ito's isometry)

By Def.

$$P_t f(x) = \mathbb{E}\left(f\left(e^{-t} x + \sqrt{1 - e^{-2t}} Z\right)\right)$$

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$$P_t f(x) = \mathbb{E}\left(f\left(e^{-t}x + \sqrt{1-e^{-2t}}\,Z\right)\right)$$

where $Z \sim N(0,1)$

$$(\mathcal{L}f)(x) = \partial_t P_t f(x)\Big|_{t=0}$$

$$= \mathbb{E}\left[f'\left(e^{-t}x + \sqrt{1-e^{-2t}}\,Z\right)\cdot\left(-e^{-t}x + \frac{e^{-2t}}{\sqrt{1-e^{-2t}}}Z\right)\right]_{t=0}$$

Note $\mathbb{E}_\mu[Z g(Z)] = \mathbb{E}_\mu[g'(Z)]$

$$\boxed{(\mathcal{L}f)(x)} = -xf'(x) + \lim_{t\to 0}\mathbb{E}\left(f'\left(e^{-t}x+\sqrt{1-e^{-2t}}\,Z\right)\cdot\frac{e^{-2t}}{\sqrt{1-e^{-2t}}}Z\right)$$

(Z')

$$= -xf'(x) + \lim_{t\to 0}\mathbb{E}\left(f''\left(e^{-t}x + \frac{1-e^{-2t}}{e^{-2t}}Z'\right)\right)$$

$$= \boxed{-xf'(x) + f''(x)}$$

$-\langle f, \mathcal{L}g\rangle$

Also $\mathcal{E}(f,g) = -\int f(x)\,\mathcal{L}g(x)\,d\mu(x)$

$$= -\int f(x)\left(g''(x) - xg'(x)\right)d\mu(x)$$

(1) integration by part

① integration by part

$$= \underset{\sim\sim\sim}{\int f'(x)\, g'(x)\, d\mu(x)} = \boxed{\mathbb{E}_\mu \left[f'(z) g'(z) \right]}$$

Since ② $\underline{\int x f(x)\, d\mu(x) = \int f'(x)\, d\mu(x)}$

In general,

$$\left.\begin{aligned} \mathcal{L}f(x) &= \Delta f(x) - x \cdot \nabla f(x) \\ \mathcal{E}(f,g) &= \mathbb{E}(\nabla f \cdot \nabla g) \end{aligned}\right\} \star$$

Thm Poincaré Inequality for O-U process

$$\underline{\mathrm{Var}_\mu(f) \leq \mathcal{E}(f,f) = \mathbb{E}_\mu\left(|\nabla f|^2\right)}$$

Pf: $\mathrm{Var}(f) = \mathrm{Cov}(f,f) \overset{\text{Lemma}}{=} \int_0^\infty \mathcal{E}(f, P_t f)\, dt$

$$= \int_0^\infty \mathbb{E}_\mu(\nabla f \cdot \nabla P_t f)\, dt$$

$$= \int_0^\infty \mathbb{E}(\nabla f \cdot \underset{\sim}{e^{-t}} P_t \nabla f)\, dt$$

CS

$$\overset{CS}{\leq} \int_0^\infty e^{-t} \left(\mathbb{E}(|\nabla f|^2) \cdot \mathbb{E}(|P_t \nabla f|^2) \right)^{1/2} dt$$

Also, by Jensen's Inequality plug in

$$\mathbb{E}(|P_t \nabla f|^2) \leq \mathbb{E}(|\nabla f|^2)$$

$$\Rightarrow \quad Var(f) \leq \underbrace{\mathbb{E}(|\nabla f|^2)}_{= \mathcal{E}(f,f)} \cdot \underbrace{\int_0^\infty e^{-t} dt}_{1}$$

$f(x) = x_1 + x_2 + \cdots x_n$ obtains the equality.

Hypercontractivity

Def: For a Markov process $(X_t)_{t\geq 0}$ with Semigroup $(P_t)_{t\geq 0}$ and equilibrium measure μ. P_t is a hypercontractivity if $\forall p>1$, $t>0$, $q=q(t,p)>p$ and $\forall f\in \mathcal{L}^p(\mu)$

$$\| P_t f \|_{\mathcal{L}^q(\mu)} \leq \|f\|_{\mathcal{L}^p(\mu)} \quad \star$$

<u>Thm</u>. hypercontractivity for the OU semigroup with $q = 1+(p-1)e^{2t}$

Lemma: (logarithmic Sobolev inequality) Let μ^n be the Standard Gaussian measure on $\mathbb{R}^n$, and $f:\mathbb{R}^n\to\mathbb{R}$ is an absolutely continuous fuction, then

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu^n} d\mu^n \le 2 \int |\nabla f|^2 d\mu^n \quad (\star)$$

Let $\nu := f^2$ $\quad \nu_t := P_t \nu \quad (\nu_t := P_t f)$

$\quad = \lim_{t\to\infty} \mathbb{E}(\nu)$

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu^n} d\mu^n = \underline{\int \nu \log \nu \, d\mu^n}$$

$$- \underline{\int \nu \, d\mu^n \cdot \log \int \nu \, d\mu^n}$$

$$= \mathbb{E}_{\mu^n}(\nu \cdot \log P_0 \nu) - \mathbb{E}_{\mu^n}(\nu) \cdot \lim_{t\to\infty} \log \nu_t$$

$$= - \int_0^\infty \partial_t \langle \nu_t , \log \nu_t \rangle dt$$

$\quad '' \, \mathbb{E}_\mu (\nu_t \log \nu_t)$

$$\overset{\text{product rule}}{=} - \int_0^\infty \langle \partial_t \nu_t , 1 + \log \nu_t \rangle dt$$

recall $\quad \partial_t \nu_t = \mathcal{L} \nu_t \quad \quad \partial_t \nu_t = \partial_t P_t \nu = \mathcal{L} P_t \nu = \mathcal{L} \nu_t$

$$= - \int_0^\infty \langle \mathcal{L} \nu_t , 1 + \log \nu_t \rangle dt$$

Note that

$$\langle \mathcal{L} \nu_t, 1+\log \nu_t \rangle =: -\mathcal{E}(\nu_t, 1+\log \nu_t)$$

$$= -\int \frac{|\nabla \nu_t|^2}{\nu_t} d\mu^n \quad (1)$$

We have $\boxed{\nabla \nu_t = e^{-t} P_t \nabla \nu}$, $\nu_t = P_t \nu$

$$|\nabla \nu_t|^2 = e^{-2t} |P_t \nabla \nu|^2 \underset{(CS)}{\leq} e^{-2t} \nu_t P_t\left(\frac{|\nabla \nu|^2}{\nu}\right) \quad (2)$$

Combining them together,

(1) and (2)

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu^n} d\mu^n \leq \int_0^\infty e^{-2t} \int P_t\left(\frac{|\nabla \nu|^2}{\nu}\right) d\mu^n dt$$

$$= \int_0^\infty e^{-2t} \int \left(\frac{|\nabla \nu|^2}{\nu}\right) d\mu^n dt \quad \left(\text{equilibrium measure}\right)$$

$$= \frac{1}{2} \int \frac{|\nabla \nu|^2}{\nu} d\mu^n = 2 \int |\nabla f|^2 d\mu^n$$

Since $\nabla \nu = 2 f \nabla f$ $(\nu = f^2)$ $\square$

Now, we prove the theorem,

Let $f_t := P_t f$ $\quad r(t) := \int f_t^{q(t)} d\mu^n = \|f_t\|_{\mathcal{L}^{q(t)}}^{q(t)}$

$$\boxed{q(t) = 1 + (p-1)e^{2t}}$$

$$\boxed{q(t) = 1 + (p-1)e^{2t}}$$

assume $f \geq 0$ everywhere

$$\underset{\text{(w.r.t } t)}{r'(t)} = \int f_t^{q(t)} \partial_t \left(q(t) \log f_t\right) d\mu^n$$

$$= q'(t) \int f_t^{q(t)} \log f_t \, d\mu^n +$$

$$q(t) \int f_t^{q(t)-1} \partial_t (f_t) \, d\mu^n$$

$\partial_t f_t$
$= \partial_t P_t f$
$= L P_t f$
$= L f_t$

$$= q'(t) \int f_t^{q(t)} \log f_t \, d\mu^n + \quad (1)$$

$$q(t) \int f_t^{q(t)-1} \cdot L f_t \, d\mu^n \quad (2)$$

$$= \underbrace{\frac{q'(t)}{q(t)} \int f_t^{q(t)} \log f_t^{q(t)} d\mu^n}_{} -$$

$$\underbrace{q(t)(q(t)-1) \int f_t^{q(t)-2} |\nabla f_t|^2 d\mu^n}_{}$$

Since $\langle L f_t, f_t^{q(t)-1} \rangle =: -\mathcal{E}(f_t, f_t^{q(t)-1})$

$$= -\int \nabla f_t \, \nabla (f_t^{q(t)-1}) \, d\mu^n$$

$$= -\int \nabla f_t \, \nabla\left(f_t^{\,q(t)-1}\right) d\mu^n$$

Power rule

$$= (q(t)-1)\int |\nabla f_t|^2 \cdot f_t^{\,q(t)-2} \, d\mu^n$$

Note $\boxed{q'(t) = 2(q(t)-1) \quad q(0)=p}$ $\left[\, q(t)-1 = \frac{q'(t)}{2} \right]$

$$\partial_t \log \|f_t\|_{L^{q(t)}(\mu^n)} = \partial_t\left(\frac{\log r(t)}{q(t)}\right)$$

Quotient

$$= \frac{-q'(t)\log r(t)}{q^2(t)} + \frac{r'(t)}{q(t)\,r(t)} \quad ③ \; ④$$

$$= \frac{q'(t)}{q^2(t)\,r(t)} \int f_t^{\,q(t)} \log \frac{f_t^{\,q(t)}}{r(t)} \, d\mu^n \quad ③$$

$$④ \quad - \frac{q(t)-1}{r(t)} \int f_t^{\,q(t)-2} |\nabla f_t|^2 \, d\mu^n$$

$$= \frac{q'(t)}{q^2(t)\,r(t)}\left(\int f_t^{\,q(t)} \log \frac{f_t^{\,q(t)}}{r(t)} \, d\mu^n - \frac{q^2(t)}{2}\int f_t^{\,q(t)-2} |\nabla f_t|^2 \, d\mu^n\right) \le 0$$

$$\int f^2 \log \frac{f^2}{\int f^2 d\mu^n} \, d\mu^n \le 2\int |\nabla f|^2 d\mu^n \quad (\star)$$

Applying logarithmic Sobolev inequality to function $f_t^{\frac{q(t)}{2}}$

$$\Rightarrow \quad \partial_t \log \|f_t\|_{L^{q(t)}(\mu^n)} \leq 0 \quad \forall t$$

$$\Rightarrow \quad \|P_t f\|_{L^{q(t)}(\mu^n)} \leq \|f\|_{L^p(\mu^n)}$$

(case $t=0$)