Subadditive Stochastic processes:

Introduce by

Hammersley and Welsh (1965).

X_{st} ($s < t$ in $\mathbb{Z}_{\geq 0}$) family of r.v. such that:

① for every $s < t < u$

$$\longrightarrow \quad X_{su} \leq X_{st} + X_{tu}$$

② The distribution of X_{st} depends only on $t - s$

③ $\mathbb{E}(X_{0t}) < +\infty$ and $\mathbb{E}(X_{0t}) \geq -At$

for some constant A and all $t \geq 1$.

Kingman strengthened condition ②

(II) the joint distribution of the process $(x_{s+1,t+1})$ are the same as those of x_{st}.

Why condition (II)

For an additive process x_{st} write

$$y_t = x_{t-1,t}$$

Then

$$\underbrace{x_{st} = y_{s+1} + y_{s+2} + \_\_ + y_t}_{\text{partial sum of seq. of r.V.}}$$

By S.L.L.N. for stationary sequences, if $\{y_t\}$ is stationary and $E(y_t) < +\infty$, then

$$\boxed{\lim_{t \to +\infty} \frac{1}{t}(y_1 + \_\_ + y_t) =} \boxed{\lim_{t \to +\infty} \frac{X_{0t}}{t}}$$

exist with prob 1.

Stationarity of $\{y_t\}$ $\Leftrightarrow$ (II) (when x_{st} additive)

Let X_{st} be a s.p. satisfying $\{(1), (II), \text{ and } (3)\}$

$$\Rightarrow \quad E(X_{st}) = g_{s-t}$$

From 1:

$$\boxed{g_{\alpha+\beta} \le g_\alpha + g_\beta} \quad (1)$$

$\{g_\alpha\}$ subadditive sequence. Then

$$\rightarrow \quad \frac{g_\alpha}{\alpha} \xrightarrow{\alpha \to \infty} \gamma \quad \text{time constant of the process } X_{st}.$$

where

$$\boxed{\gamma = \inf_{\alpha \geq 1} \frac{g_\alpha}{\alpha}}.$$

Theorem (Kingman's Subadditive Ergodic theorem)

Let x_{st} be a stationary subadditive process with time constant γ then

$$\lim_{t \to \infty} \frac{x_{0t}}{t}$$ ①

exist almost surely. If we write $\xi = \lim \frac{x_{0t}}{t}$ then

$$\mathbb{E}(\xi) = \gamma.$$ ②

Also

$$\mathbb{E}\left| \frac{x_{0t}}{t} - \xi \right| \xrightarrow{t \to \infty} 0$$ ③

thus $x_{0t}/t \longrightarrow \xi$ in prob.

Moreover if $\mathcal{I}$ is the σ-alg. of events defined in terms of (x_{st}) and invariants under the map

$$x_{st} \overset{T}{\longmapsto} x_{s+1, t+1}$$

then

$$\xi = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}(X_{0t} \mid \mathcal{I}). \quad (4)$$

In particular if $\mathcal{I}$ is trivial (T ergodic) then

$$\xi = \gamma.$$

(*) the proof of (1) depend on a decomposition

$$X_{st} = Y_{st} + Z_{st}$$

with Y_{st} additive process, $\mathbb{E}(Y_{01}) = \gamma$ and Z_{st} nonnegative subadditive process with time constant $= 0$.

Ω : $\Theta(s,t) \leq \Theta(s,u) + \Theta(u,t)$

Organization of the proof :

(i) $\xi^+ = \limsup_{t\to\infty} \frac{X_{0t}}{t}$ is a.s. finite.

(ii) $\mathbb{E}(\xi^+) = \gamma$

(iii) $\mathbb{E}\left(\frac{X_{0t}}{t} - \xi^+\right) \longrightarrow 0$

(iv) $\xi^+ = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}(X_{0t} \mid \mathcal{I})$.

(v) Use ⊛ to conclude that

$$\xi^+ = \xi = \lim_{t\to\infty} \frac{1}{t} X_{0t}$$

$\liminf \frac{1}{t} X_{0t}$

Proof:

(i) Fix n ; $\boxed{N(t) = \lfloor t/n \rfloor}$

$T(0, nx)$

$t \quad (n-1)N(t) + n-1$

$$x_{0t} \le \underbrace{\sum_{r=1}^{N(t)} x_{(r-1)n,\,rn}} + x_{N(t)n,\,t}$$

$$\le \sum_{r=1}^{N(t)} x_{(r-1)n,\,rn} + \underbrace{\sum_{u=0}^{n-1} |X_{N(t)n,\,N(t)n+u}|}_{w_{N(t)}}$$

γ

0

- $\underline{(x_{(r-1)n,\,rn})}$ stationary with finite expectation g_n

By S.L.L.N the following limit exist a.s.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} x_{(t-1)n,\, tn} = \boxed{z_n}.$$

and $\boxed{\mathbb{E}(z_n) = g_n.}$ (*)

For each $\varepsilon > 0$

$$\sum_{N=1}^{\infty} \underbrace{P(w_N / N \geq \varepsilon)}\ \overset{\mathrm{II}}{=} \sum_{N=1}^{\infty} P(w_0 \geq N\varepsilon)$$

$$\leq \underbrace{\frac{1}{\varepsilon} \mathbb{E}(w_0) . < +\infty}_{}$$ (3) (*)

By Borel - Cantelli

$$\longrightarrow \underbrace{\lim_{N\to\infty} w_N / N = 0}_{} \quad \text{a.s.}$$

$$\Rightarrow \xi^{+} \leq \limsup_{t\to\infty} \frac{x_{0t}}{N(t)n}$$

$$\leq \underbrace{\limsup_{t\to\infty} \overbrace{\frac{1}{N(t)n} \left[\sum_{t=1}^{N(t)} x_{(t-1)n,\, tn}\right.}^{z_n} \overbrace{\left. + w_{N(t)} \right]}^{0}}_{}$$

$$= \frac{z_n}{n}. \longrightarrow \boxed{\xi^{+} \leq \frac{z_n}{n}.}$$

$$\Rightarrow \quad \xi^+ < +\infty \quad \text{a.s.} \quad \text{and} \quad \underbrace{\mathbb{E}(\xi^+) \le g_n/n}_{\text{for every } n}$$

$= \mathbb{E}(\mathfrak{z}_n)$

$$\Rightarrow \quad \mathbb{E}(\xi^+) \le \gamma.$$

To prove $\mathbb{E}(\xi^+) \ge \gamma$

$$\cdot \quad a_{st} = \sum_{j=s+1}^{t} x_{j-1,j}$$

a_{st} is additive and $x_{st} \le a_{st}$

$$\cdot \quad b_{st} = a_{st} - x_{st}$$

Non negative superadditive

$$\cdot \quad B_n = \inf_{t \ge n} b_{0t}/t \quad \rightarrow \quad \text{Increasing}$$

$$\lim_{t\to\infty} \inf b_{0t}/t = \lim_{t\to\infty} \inf \frac{1}{t}(a_{0t} - x_{0t})$$

$$= \mathfrak{z}_1 - \xi^+$$

By monotone convergence

$$\lim_{n\to\infty} \mathbb{E}(B_n) = \mathbb{E}(\lim B_n)$$

$$= \mathbb{E}(g_t - \xi^+)$$

$$= g_t - \mathbb{E}(\xi^+)$$

Also

$$\lim_{n\to\infty} \mathbb{E}(B_n) \leq \lim_{n\to\infty} \mathbb{E}\left(b_{0n}/n\right)$$

$\inf \frac{b_{0n}}{n}$ $\quad$ a_{0n} $\quad$ X_{0n}

$$= \lim_{n\to\infty} \frac{1}{n}(n g_t - g_n)$$

$$= g_t - \gamma.$$

$$\Rightarrow \quad \boxed{g_t - \mathbb{E}(\xi^+) \leq g_t - \gamma.}$$

$\mathbb{E}(\xi^+) = \gamma.$

This proves (ii).

$$\boxed{B_t = \inf_{\alpha \geq t} \frac{b_{0\alpha}}{\alpha}.}$$

Now

$$\mathbb{E}\left|\frac{1}{t} b_{0t} - B_t\right| = \mathbb{E}\left(\frac{1}{t} b_{0t} - B_t\right)$$

$$= \underbrace{\left(\underbrace{g_\delta}_{\frac{1}{t}\mathbb{E}(a_{0t})} - \underbrace{g_t/t}_{\mathbb{E}(x_{0t})} \right)}_{\longrightarrow\, g_\delta - \gamma} - \mathbb{E}(B_t) \longrightarrow g_\delta - \gamma$$

$$\Rightarrow \mathbb{E}\left| \underbrace{\tfrac{1}{t} b_{0t} - B_t} \right| \longrightarrow 0.$$

B_t monotone and $\underbrace{\mathbb{E}(B_t) \longrightarrow g_\delta - \gamma}$

$$\Rightarrow t^{-1} b_{0t} \longrightarrow \underbrace{\zeta_\delta - \xi^+}$$

By the Ergodic theorem.

$$\frac{1}{t} a_{0t} \longrightarrow \zeta_\delta$$

$$\Rightarrow \frac{1}{t} x_{0t} = \underbrace{\frac{1}{t}(\overset{\zeta_\delta}{a_{0t}} - \overset{\zeta_\delta - \xi^+}{b_{0t}})} = \xi^+$$

in mean and also in probability.

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(N) $x_{st} = y_{st} + z_{st} \longrightarrow$

Sta. add. $\mathbb{E}(y_{0t}) = \gamma$.

$\xi = \limsup_{t \to \infty} z_{0t}/t$; $\mathbb{E}(\xi) = 0$

$$\boxed{\lim_{t \to \infty} z_{0t}/t = 0.}$$ (i)

$\rightarrow \lim_{t \to \infty} y_{0t}/t$ Exist

$$\lim \frac{x_{0t}}{t} = \lim \frac{y_{0t}}{t} + \lim \frac{z_{0t}}{t}$$

$\parallel$

ξ^{+}