Math 210, Fall 2025

Problem Set # 1

Question 1. A fair coin is flipped 7 times. Let X = total number of heads. The sample space consists of all possible sequences of outcomes of the 7 flips.

a) Write down three possible outcomes ω from the sample space Ω.

Solution. Three possible outcomes are

$$(H,H,H,H,H,H,H),(T,H,H,H,H,H,H),(T,T,H,H,H,H,H).$$

b) How many outcomes are in the sample space?

Solution. The sample space has 2^7 different outcomes.

c) Compute $P(\{\omega\})$ for each outcome ω you wrote down in part (a).

Solution. All outcomes are equally likely. The probability of each outcome is $1/2^7$.

d) Compute $X(\omega)$ for each outcome ω you wrote down in part (a).

X counts the total number of heads.

$$X(H,H,H,H,H,H,H) = 7$$
$$X(T,H,H,H,H,H,H) = 6$$
$$X(T,T,H,H,H,H,H) = 5.$$

e) Find $P(X \leq 5)$. Hint: Find $P(X > 5)$ first.

Solution. Using the hint we find

$$P(X \leq 5) = 1 - P(X > 5) = 1 - P(X = 6) - P(X = 7).$$

Now computing each probability

$$P(X = 7) = \frac{1}{2^7}$$
$$P(X = 6) = \frac{7}{2^7}$$

thus

$$P(X \leq 5) = 1 - \frac{1}{2^7} - \frac{7}{2^7} = \frac{2^7 - 8}{2^7}.$$

Question 2. Consider a coin where the probability of heads is $0 < p < 1$. Do **not** assume $p = 1/2$. Flip it until the first tails occurs. Let X = number of flips needed to see the first tails.

1. How many possible outcomes are in the sample space?

   Solution. There are infinitely many outcomes.

2. Find $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$.

   Solution.

$$
\begin{aligned}
P(X = 1) &= (1 - p) \\
P(X = 2) &= p(1 - p) \\
P(X = 2) &= p^2(1 - p).
\end{aligned}
$$

3. Write down a formula for $P(X = k)$, where k is a positive integer.

   Solution.

$$P(X = k) = p^{k-1}(1 - p).$$

   this is an example of the geometric distribution.

Question 3. In E.O. Thorp's *The Mathematics of Gambling* he considers a strategy for blackjack due to Baldwin et al in which the probability that the player wins (and the house dealer loses) is .5005. Suppose this is accurate. If you have \$100 to gamble and on one turn you gamble $100y$ dollars where y is a number in $[0, 1]$ then after one turn your resulting fortune is

$$
\begin{cases}
100(1 + y) & \text{with probability } 0.5005 \\
100(1 - y) & \text{with probability } 0.4995.
\end{cases}
$$

Let X be the random variable representing your fortune after one turn of the game.

1. Find the expectation $E[X]$ as a function of y.

   Solution.

   $E[X] = 100(1 + y)P(X = 100(1 + y)) + 100(1 - y)P(X = 100(1 - y)) = 100 + 0.1y.$

2. Find the value of y in $[0, 1]$ which maximizes $E[X]$.

Solution. $E[X]$ is a line with positive slope and its maximum value occurs at $y = 1$.

3. Consider the random variable

$$k = \log(X/100)$$

known as the *logarithmic return rate*. Calculate $E[k]$ as a function of y.

Solution.

$$\begin{aligned} E[k] &= \log(100(1+y)/100)P(X = 100(1+y)) + \log(100(1-y)/100)P(X = 100(1-y)) \\ &= 0.5005 \log(1+y) + 0.4995 \log(1-y). \end{aligned}$$

4. Find the value of y in $[0, 1]$ which maximizes $E[k]$.

Solution. First we find the critical points

$$\frac{d}{dy}E[k] = \frac{0.5005}{y+1} - \frac{0.4995}{1-y}.$$

now setting $\frac{d}{dy}E[k] = 0$ we find $y = .001$. Now we evaluate $E[k]$ at $0.001, 0$ and 1. The maximum value of $5 \cdot 10^{-7}$ is obtained at the point $y = 0.001$.