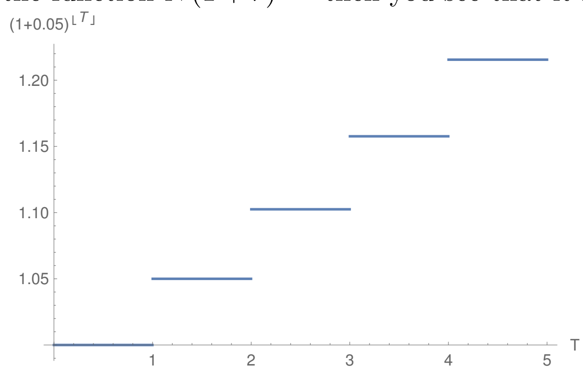
Math 210, Fall 2025

Problem Set # 2

Question 1. Suppose the compound interest function for borrowing and lending from banks is $N(1+r)^{\lfloor T \rfloor}$, where N is the principal, r is the rate of interest, and $T \in \mathbb{R}$ is the time period. Construct an arbitrage portfolio.

Solution. If you plot the function $N(1+r)^{\lfloor T \rfloor}$ then you see that it is a step function that



jumps at the integers.

So to exploit this discontinuity, you have to return what you owe before the function jumps, but keep what you've invested in until *after* the function jumps.

1. At time 0, borrow 1 from bank 1 for a term of 1 year. Invest 1 in bank 2 for a term of 0.9 years.
2. At time 0.9, you owe $1(1+r)^{\lfloor 0.9 \rfloor} = 1(1+r)^0 = 1$ to bank 1. So borrow 1 from bank three, and return it to bank 1.
3. At time 1, withdraw your investment from bank 2, which is worth $(1+r)^{\lfloor 1 \rfloor} = 1 + r$. Return 1 to bank 3, which pays off your loan since you only owe $(1+r)\lfloor 1 - 0.9 \rfloor = 1$.
4. You are left with a guaranteed profit of r.

Arbitrage portfolios always follow the following format.

1. Start with a portfolio worth 0
2. ???
3. Profit

Question 2. Consider a class of 50 students. For each student a fair six-sided die will be rolled to determine the student's final grade. If the die shows 6 the grade is 90. If the die shows any other number the grade is 40. Let X_i be the grade of the i-th student. Let $Z = \frac{1}{50}\sum_{i=1}^{50} X_i$ be the class average.

a) What is the expected grade of the i-th student?

Solution.

$$E[X_i] = \frac{1}{6}90 + \frac{5}{6}40 = \frac{145}{3} = 48.33.$$

b) If only 8 students roll a 6, what is the class average?

Solution.

$$\text{average} = \frac{8}{50} \cdot 90 + \frac{50-8}{50} \cdot 40 = 48.$$

c) What is the expected class average?

Solution.

$$E[Z] = \frac{1}{50} \sum_{i=1}^{50} E[X_i] = E[X_1] = 48.33.$$

Question 3. Consider a coin where the probability of heads is p. Flip the coin n times. Define $X_i = 1$ if the i-th flip is heads, 0 otherwise. Define $Y = \sum_{i=1}^{n} X_i$.

a) Express the event that there are exactly k heads in terms of Y. Hint: If there are exactly k heads, what is the value of Y?

Solution. The event that there are exactly k heads is the event $Y = k$.

b) Find $\mathbb{E}(Y)$.

Solution.

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\ &= n\,\mathbb{E}(X_1) \\ &= np.\end{aligned}$$

c) Evaluate $\mathbb{E}(Y)$ in the case $p = 1/2$. Does your answer make sense? Why or why not?

Solution. In this case we get $\mathbb{E}(Y) = n/2$. This makes sense since we would expect to get roughly half heads and half tails.

Question 4. The variance of a random variable X is

$$\mathrm{Var}(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right).$$

It is the average squared-distance between X and its average $\mathbb{E}(X)$.

a) Use the properties of expectation to show that $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$.

Solution.

$$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}\left((X - \mathbb{E}(X))^2\right) \\ &= \mathbb{E}\left(X^2 - 2X\,\mathbb{E}(X) + \mathbb{E}(X)^2\right) \\ &= \mathbb{E}(X^2) - 2\,\mathbb{E}(X)\,\mathbb{E}(X) + \mathbb{E}(X)^2 \\ &= \mathbb{E}(X^2) - \mathbb{E}(X)^2.\end{aligned}$$

b) Let X be the number shown after rolling a fair six-sided die. Find $\mathbb{E}(X^2)$ and use it to compute $\mathrm{Var}(X)$.

Solution.

$$\mathbb{E}(X) = \sum_{i=1}^{6} \frac{1}{6} i = \frac{1}{6} \sum_{i=1}^{6} = \frac{21}{6} = \frac{7}{2}.$$

$$\mathbb{E}(X^2) = \sum_{i=1}^{6} \frac{1}{6} i^2 = \frac{1}{6} \sum_{i=1}^{6} i^2 = \frac{91}{6}.$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{35}{12}.$$

Question 5. Two market standards for US dollar interest rates are semi-annual compounding with 30/360 daycount (semi-bond, denoted y_{sb}) and annual compounding with act/360 daycount (annual-money, denoted y_{am}).

a) Derive an expression for y_{am} in terms of y_{sb}. You may assume all years have 365 days.

Solution. Both interest rates must give the same accrued interest for one year. This implies

$$\left(1 + y_{am}\frac{365}{360}\right) = \left(1 + y_{sb}\frac{180}{360}\right)^2 = \left(1 + y_{sb}\frac{1}{2}\right)^2.$$

Solving for Y_{am} we find.

$$y_{am} = \left(y_{sb} + \frac{y_{sb}^2}{4}\right) - \frac{1}{73}.$$

b) Calculate y_{am} when $y_{sb} = 0.5, 0.6$, and 0.7.

Solution.

$$y_{am}(0.5) = 0.554795$$
$$y_{am}(0.6) = 0.680548$$
$$y_{am}(0.7) = 0.811233$$

Question 6. A *simple interest rate* of r for T years means a 100 investment becomes $100(1 + rT)$ at maturity T.

a) For simple interest 5% for ten years, calculate the equivalent interest rate with (i) annual, (ii) quarterly and (iii) continuous compounding. Assume 30/360 daycount.

Solution. Let $q = 5/100 = .05$.

(i) To find the annual rate r we solve

$$1 + qT = \left(1 + \frac{360}{360} r\right)^T$$

for r and find

$$r = (1 + qT)^{1/T} - 1.$$

Plugging in values we get $r = .0414..$

(ii) To find the quarterly rate r we solve

$$1 + qT = \left(1 + \frac{90}{360} r\right)^{4T}$$

for r and find

$$r = 4\left((1 + qT)^{1/(4T)} - 1\right).$$

Plugging in values we get $r = .0408..$

(ii) To find the continuous rate r we solve

$$1 + qT = e^{rT}$$

for r and find

$$r = \frac{1}{T} \ln(1 + qT).$$

Plugging in values we get $r = .0405..$

b) Show that if simple interest of r for T years is equivalent to r^* interest rate with annual compounding, then $r^* \to 0$ as $T \to \infty$.

Solution. We have the equation

$$1 + rT = (1 + r^*)^T.$$

We can solve this equation for r^* to find

$$r^* = -1 + (1 + rT)^{1/T}.$$

Now taking limits we get

$$\begin{aligned}\lim_{T\to\infty} r^* &= \lim_{T\to\infty} -1 + (1 + rT)^{1/T} \\ &= -1 + \lim_{T\to\infty} (1 + rT)^{1/T} \\ &= -1 + 1 = 0.\end{aligned}$$

To evaluate the limit we use L'Hospital's Rule as follows

$$\begin{aligned}\lim_{T\to\infty} (1 + rT)^{1/T} &= \lim_{T\to\infty} e^{\frac{1}{T}\ln(1+rT)} \\ &= \exp\left(\lim_{T\to\infty} \frac{1}{T}\ln(1 + rT)\right) \\ &= \exp\left(\lim_{T\to\infty} \frac{1}{1}\frac{1}{1 + rT}r\right) \\ &= e^0 = 1.\end{aligned}$$

Question 7. Suppose annually compounded zero rates for all maturities (with 30/360 daycount) are r. An annuity pays n at years $n = 1, 2, \ldots, N$.

1. What is the present value of the annuity? *Hint:*

$$\frac{d}{dr}\sum_{n=1}^{N} \frac{1}{(1 + r)^n} = -\frac{1}{1 + r}\sum_{n=1}^{N} \frac{n}{(1 + r)^n}.$$

Solution. The zero rate is r which implies $Z(t, T) = e^{-r(T-t)}$. Now we can find the value V of the annuity as follows

$$
\begin{aligned}
V &= \sum_{n=1}^{N} nZ(0,n) \\
&= \sum_{n=1}^{N} \frac{n}{(1+r)^n} \\
&= -(1+r)\frac{d}{dr}\sum_{n=1}^{N} \frac{1}{(1+r)^n} \\
&= -(1+r)\frac{d}{dr}\left(\frac{1}{r}\left(1-\frac{1}{(1+r)^N}\right)\right) \\
&= \frac{1+r}{r^2} - \frac{1+r+Nr}{r^2(1+r)^N}.
\end{aligned}
$$

2. What is the present value of the infinite annuity as $N \to \infty$?

   Solution.

$$
\begin{aligned}
\lim_{N\to\infty} V &= \lim_{N\to\infty} \frac{1+r}{r^2} - \frac{1+r+Nr}{r^2(1+r)^N} \\
&= \frac{1+r}{r^2}.
\end{aligned}
$$