Math 210, Fall 2025

Problem Set # 3 Solutions

Question 1. Equivalent interest rates and doubling time.

a) Interest is compounded twice per year for 5 years at a rate of $r_s = 3\%$ per annum. Find the equivalent interest rate r_A for annual compounding and r for continuous compounding. Assume 30/360 daycount.

b) Find the number of years required for your balance to double if interest is compounded annually at rate r_A, twice per year at rate r_s and continuously at rate r. Give your answer correct to within one-tenth of a year.

c) Find the doubling using the Rule of 72. How does this compare with your answer from part b?

d) Consider an asset that pays N at maturity 5 years from now. If the present value of the asset is \$300 find N. Use the interest rates from part (a) to compute N.

Solution. a) Annual compounding:

$$(1 + r_s/2)^{2\cdot 5} = (1 + r_A)^5$$

$$r_A = (1 + r_s/s)^2 - 1 = 0.030225.$$

Continuous compounding:

$$(1 + r_s/2)^{2\cdot 5} = e^{r\cdot 5}$$

$$r = 2\ln(1 + r_s/2) = 0.029777.$$

b) Annual compounding:

$$\begin{aligned} 2 &= (1 + r_A)^T \\ T &= \frac{\ln(2)}{\ln(1 + r_A)} \\ &= 23.28. \end{aligned}$$

Continuous compounding:

$$\begin{aligned} 2 &= e^{rT} \\ T &= \frac{\ln(2)}{r} \\ &= 23.28. \end{aligned}$$

c)

$$\frac{72}{3} = 24$$

d)

$$N = 300e^{r \cdot 5} = \$348.16$$

Question 2. Consider an annuity starting at time 0 that pays 1 each year for M years. Assume the annually compounded zero rate is r_A for all maturities $T = 1, \ldots, M$. In an example in class, we found that the value of this annuity at its starting time 0 is

$$V_0 = \sum_{i=1}^{M} Z(0,i) = \frac{1-(1+r_A)^{-M}}{r_A}.$$

Find the value of the annuity at time t, where $0 < t < 1$.

Solution.

$$\begin{aligned} V(t) &= \sum_{i=1}^{M} Z(t,i) \\ &= \sum_{i=1}^{M} \frac{1}{(1+r_A)^{i-t}} \\ &= \frac{1}{(1+r_A)^{-t}} \sum_{i=1}^{M} \frac{1}{(1+r_A)^{t}} \\ &= (1+r_A)^t \left(\frac{1-(1+r_A)^{-M}}{r_A} \right) \\ &= (1+r_A)^t V(0). \end{aligned}$$

Question 3. (a) Consider an annuity that pays 1 every quarter for M years. In other words, the payment times are $T = t + \frac{1}{4}, t + \frac{2}{4}, \ldots, t + \frac{4M}{4}$. Show that the value at present time t is

$$V_t = \frac{1-(1+r_4/4)^{-4M}}{r_4/4},$$

assuming the quarterly compounded interest rate has constant value r_4.

(b) A **fixed rate bond** with notional N, coupon c, start date T_0, maturity T_n, and term length α is an asset that pays N at time T_n and coupon payments $\alpha N c$ at times T_i for $i = 1, \ldots, n$, where $T_{i+1} = T_i + \alpha$. It is equivalent to an annuity plus N ZCBs. Consider a fixed rate bond with notional N and coupon c that starts now, matures M years from now, and has quarterly coupon payments. Show that the value at present time $t = 0$ is

$$V_t = \frac{cN}{4} \cdot \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4} + N(1 + r_4/4)^{-4M},$$

assuming the quarterly compounded interest rate has constant value r_4.

Solution. a)

$$\begin{aligned}
V(0) &= \sum_{i=1}^{M} (Z(0,i) + Z(0,i+1/4) + Z(0,i+2/4) + Z(0,i+3/4)) \\
&= \sum_{i=1}^{M} \frac{1}{(1 + r_4/4)^{4i}} + \frac{1}{(1 + r_4/4)^{4i+1}} + \frac{1}{(1 + r_4/4)^{4i+2}} + \frac{1}{(1 + r_4/4)^{4i+3}} \\
&= \sum_{i=1}^{4M} \frac{1}{(1 + r_4/4)^{4i}} \\
&= \frac{(1 + r_4/4)^{-4M+1} - (1 + r_4/4)^{-1}}{1 - (1 + r_4/4)^{-1}} \\
&= \frac{(1 + r_4/4)^{-4M+1} - (1 + r_4/4)^{-1}}{-\frac{r_4/4}{1+r_4/4}} \\
&= -(1 + r_4/4)^{-1} \left(\frac{(1 + r_4/4)^{-4M+1} - (1 + r_4/4)^{-1}}{r_4/4} \right) \\
&= \frac{1 - (1 + r_4/4)^{-4M}}{r_4/4}.
\end{aligned}$$

b) Quarterly payments so $\alpha = 4$.

$$\begin{aligned}
V(0) &= Z(0,M)N + \frac{Nc}{4} \sum_{i=1}^{M} (Z(0,i) + Z(0,i+1/4) + Z(0,i+2/4) + Z(0,i+3/4)) \\
&= \frac{N}{(1 + r_4/4)^{4M}} + \frac{Nc}{4} \left(\frac{1 - (1 + r_4/4)^{-4M}}{r_4/4} \right).
\end{aligned}$$

Question 4. The current US Dollar (USD) to Japense Yen (JPY) exchange rate is 0.00925USD/JPY.

a) Find the JPY to USD exchange rate.

b) Find the value in USD of 300,000 JPY

Solution. a)

$$\frac{1}{0.00925} = 108.108 \text{ JPY/USD}$$

b)

$$300000 \cdot .00925 = 2775 \text{ USD}.$$