Math 210, Spring 2020

Problem Set # 4

Due February 14, 2020 at 5pm on gradescope

Question 1. Consider a fixed rate bond with notional N, coupon c, start date T_0 and maturity 10 years which makes semi-annual coupon payments. This bond pays $Nc/2$ semi-annually and notional N at maturity. The **price** P of a bond is the present value of its cash flow. The **yield to maturity (YTM)** at T_0 of this bond is the value of r such that

$$P = N\frac{c}{2}\sum_{i=1}^{20}\frac{1}{(1+r/2)^{2i-1}} + \frac{N}{(1+r/2)^{2n}},$$

that is, the rate at which discounting the remaining payments gives the bond price.

a) FORD MOTOR CREDIT CO LLC issued bonds on Sept. 9, 2014 with notional \$100, maturity Sept. 2024, coupon 3.664%, and semi-annual coupon payments. Suppose the YTM of the bond on Sept. 19, 2019 is 3.812%. What is the price of the bond today? You may assume there are 10 coupon payments remaining.

b) Suppose the bond is currently trading at a price of \$98.00. What is the YTM given this new price?

c) If a bond's price rises will its yield increase or decrease? Explain why bond yields and prices move in opposite directions.

Solution. a) Since there are 5 years remaining we have

$$\begin{aligned}
P &= \frac{Nc}{2}\sum_{i=1}^{10}\frac{1}{(1+r/2)^{i}} + \frac{N}{(1+r/2)^{10}} \\
&= \frac{Nc}{2}\frac{1-(1+r/2)^{-10}}{r/2} + \frac{N}{(1+r/2)^{10}} \\
&= 99.332.
\end{aligned}$$

b) We need to solve the equation

$$P = \frac{Nc}{2}\frac{1-(1+r/2)^{-10}}{r/2} + \frac{N}{(1+r/2)^{10}}$$

for r. We can do this in Mathematica with the following code:

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Solve[Evaluate[
  100 coup /2 Sum[(1 + r/2)^(-i), {i, 1, 10}] +
    100 (1 + r/2)^(-10) == pr /. {coup -> 3.664/100, pr -> 98.0}] &&
  r > 0, r]
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We find $r = 4.11\%$.

c) If a bond's price rises its yield will decrease. The price of a bond is the value of the cash flow received by the bondholder discounted to today. If the yield increases then this discount factor increases and thus the price decreases.

Question 2. At current time t a stock paying no income has price 45, the forward price with maturity T on the stock is 40, and the price of a zero coupon bond with maturity T is 0.95. Determine whether there is an arbitrage opportunity. If there is, find an arbitrage portfolio. Verify the portfolio you construct is an arbitrage portfolio.

Solution. For a stock paying no income we have $F(t,T) = S_t e^{r(T-t)}$. In terms of ZCB prices this gives $F(t,T) = S_t/Z(t,T)$. For this problem we have $S_t/Z(t,T) = 45/0.95 = 47.37$. This means that the forward contract is cheap. At time t we go long a forward contract at the forward price, short 1 share of stock, and buy $S_t/Z(t,T)$ ZCB's. The transactions are shown in the table below:

Portfolio	time t	time T
1 long forward contract at forward price	0	$S_T - F(t,T) = S_T - 40$
short 1 share stock	$-S_t = -45$	$-S_T$
$S_t/Z(t,T)$ ZCB's	$\frac{S_t}{Z(t,T)} Z(t,T) = S_t = 45$	$\frac{S_t}{Z(t,T)} = 45/0.95 = 47.37$
Value	0	7.37

Since the value at time t is zero and the value at time T is positive this is an arbitrage portfolio.

Question 3. At time t you own one stock that pays no dividends, and observe that $F(t,T) < S_t/Z(t,T)$. What arbitrage is available to you, assuming that you can only trade the stock, ZCB and forward contract? Be precise about the transactions you should execute to exploit the arbitrage.

Solution. This problem is very similar to question 2. At time t we go long a forward contract at the forward price, short 1 share of stock, and buy $S_t/Z(t,T)$ ZCB's. The transactions are shown in the table below: Here is the corresponding portfolio:

Portfolio	time t	time T
1 long forward contract at forward price	0	$S_T - F(t,T) = S_T$
short 1 share stock	$-S_t$	$-S_T$
$S_t/Z(t,T)$ ZCB's	$\frac{S_t}{Z(t,T)} Z(t,T) = S_t$	$\frac{S_t}{Z(t,T)}$
Value	0	$S_t/Z(t,T) - F(t,T)$

Since $F(t,T) < S_t/Z(t,T)$ the value at time T is positive and this is an arbitrage portfolio.

Question 4. Let $V_K(t,T)$ be the value of a forward contract on an asset with delivery price K,

$$V_K(t,T) = (F(t,T) - K)e^{-r(T-t)}.$$

a) Verify that $V_K(T,T)$ equals the payout of a forward contract with delivery price K. For an asset that pays no income, substitute the expression for its forward price into the above equation and give an intuitive explanation for the resulting expression.

b) Suppose at time t_0 you go short a forward contract on an asset that pays no income with maturity T (and with delivery price equal to the forward price). At time t, $t_0 < t < T$, suppose both the price of the asset and interest rates are unchanged. How much money have you made or lost? This is sometimes called the **carry** of the trade.

Solution. a) At time T we have $F(T,T) = S_T$ and so

$$V_K(T,T) = (F(T,T) - K)e^{-r(T-T)} = F(T,T) - K = S_T - K$$

which is the payout of a forward contract with delivery price K.

For an asset that pays no income, $F(t,T) = S_t e^{r(T-t)}$ and so we get

$$V_{F(t,T)} = (F(t,T) - K)e^{-r(T-t)} = (S_t e^{r(T-t)} - K)e^{-r(T-t)} = S_t - Ke^{-r(T-t)}.$$

This is the difference between the spot price of the asset and the delivery price discounted to today.

b)

$$\begin{aligned}V_{F(t_0,T)}(t,T) &= (F(t,T) - F(t_0,T))e^{-r(T-t)} \\ &= (S_t e^{r(T-t)} - S_{t_0} e^{r(T-t_0)})e^{-r(T-t)} \\ &= S_t(1 - e^{r(t-t_0)}).\end{aligned}$$

If you short the forward contract then you have made an amount of

$$-S_t(1 - e^{r(t-t_0)}) = S_t(e^{r(t-t_0)} - 1).$$

Note that $e^{r(t-t_0)} > 1$ so you make money in this situation.

Question 5. The current price of a stock paying no income is 30. Assume the annually compounded zero rate will be 3% for the next 2 years.
(a) Find the current value of a forward contract on the stock if the delivery price is 25 and maturity is in 2 years.
(b) If the stock has price 35 at maturity, find the value of the forward from part (a) to the long counterparty at maturity.

Solution. a) Using annually compounded interest we get

$$\begin{aligned}V_K(t,T) &= (F(t,T) - K)(1+r)^{-(T-t)} \\ &= (S_t(1+r)^{T-t} - K)(1+r)^{-(T-t)} \\ &= 6.44.\end{aligned}$$

b) At maturity the value to the long counterparty is

$$S_T - K = 35 - 25 = 10.$$