Math 210, Fall 2025

Problem Set # 5

Question 1. Let S_t be the current price of a stock that pays no dividends.

a) Let r_{bid} be the interest rate at which one can invest/lend money, and r_{off} be the interest rate at which one can borrow money, $r_{\text{bid}} \leq r_{\text{off}}$. Both rates are continuously compounded. Using arbitrage arguments, find upper and lower bounds for the forward price of the stock for a forward contract with maturity $T > t$.

b) How does your answer change if the stock itself has bid price $S_{t,\text{bid}}$ and offer price $S_{t,\text{off}}$?

Solution. a) We guess

$$S_t e^{r_{\text{bid}}(T-t)} \leq F(t,T) \leq S_t e^{r_{\text{off}}(T-t)}$$

and show this using arbitrage arguments. First suppose $F(t,T) > S_t e^{r_{\text{off}}(T-t)}$.. In this case the forward price is expensive so we should short the forward contract. Consider the following portfolio:

Portfolio	time t	time T
1 short forward contract at forward price	0	$F(t,T) - S_T$
long 1 share stock	S_t	S_T
cash	$-S_t$	$-S_t e^{r_{\text{off}}(T-t)}$
Value	0	$F(t,T) - S_t e^{r_{\text{off}}(T-t)}$

By assumption $F(t,T) - S_t e^{r_{\text{off}}(T-t)} > 0$ and thus this is an arbitrage portfolio. None exist and so we must have $F(t,T) \leq S_t e^{r_{\text{off}}(T-t)}$.

Now assume $F(t,T) < S_t e^{r_{\text{bid}}(T-t)}$. In this case the forward price is cheap so we should go long a forward contract. Consider the following portfolio:

Portfolio	time t	time T
1 long forward contract at forward price	0	$S_T - F(t,T)$
short 1 share stock	$-S_t$	$-S_T$
cash	S_t	$-S_t e^{r_{\text{bid}}(T-t)}$
Value	0	$-F(t,T) + S_t e^{r_{\text{bid}}(T-t)}$

By assumption $-F(t,T) + S_t e^{r_{\text{bid}}(T-t)} < 0$ and thus this is an arbitrage portfolio. None exist and so we must have $F(t,T) \geq S_t e^{r_{\text{bid}}(T-t)}$.

b) The bid price $S_{t,\text{bid}}$ is the price at which the exchange buys. The offer price $S_{t,\text{off}}$ is the price at which the exchange sells. For stocks we have $S_{t,\text{bid}} \leq S_{t,\text{off}}$. In the first portfolio above you are buying from the exchange and pay the price $S_{t,\text{off}}$. Plugging that value in we find

$$F(t,T) \leq S_{t,\text{off}} e^{r_{\text{off}}(T-t)}.$$

In the second portfolio you are selling stock to the exchange and you receive $S_{t,\text{bid}}$. Plugging in that value we find

$$S_{t,\text{bid}} e^{r_{\text{bid}}(T-t)} \leq F(t,T).$$

Putting both inequalities together we conclude

$$S_{t,\text{bid}} e^{r_{\text{bid}}(T-t)} \leq F(t,T) \leq S_{t,\text{off}} e^{r_{\text{off}}(T-t)}.$$

Question 2. FX forwards are among the most liquid derivative contracts in the world and often reveal more about the health of money markets (markets for borrowing or lending cash) than published short-term interest rates themselves.

a) On Oct. 3, 2008 the euro dollar FX rate was trading at €1 = \$1.3772, and the forward price for an April 3, 2009 forward contract was \$1.3891. Assuming six-month euro interest rates were 5.415%, what is the implied six-month dollar rate? Both interest rates are quoted with act/360 daycount and semi-annual compounding. There are 182 days between Oct. 3, 2008 and April 3, 2009.

b) Published six-month dollar rates were actually 4.13125%. What arbitrage opportunity existed? What transactions does a potential arbitrageur need to to undertake to exploit this opportunity?

c) During the financial crisis, several European commercial banks badly needed to borrow dollar cash, but their only source of funds was euro cash from the European Central Bank (ECB). These banks would: borrow euro cash for six months from the ECB; sell euros/buy dollars in the spot FX market; and sell dollars/buy euros six months forward (to neutralize the FX risk on their euro liability). Explain briefly how these actions may have created the arbitrage opportunity in (b), which existed for several months in late 2008.

Solution. a) In class we found that the forward price for foreign exchange was $F(t,T) = X_t e^{(r_\$ - r_f)(T-t)}$ for interest compounded continuously. If interest is compounded annually m times per year with accrual factor α then we get $F(t,T) = X_t \left(\frac{1+\alpha r_\$}{1+\alpha r_f}\right)^{m(T-t)}$. For this problem interest is compounded once and $\alpha = 182/360$ so we must solve

$$F(0,1/2) = X_0 \left(\frac{1+\alpha r_\$}{1+\alpha r_f}\right)$$

for r_f. We find $r_\$ = 7.17\%$.

b) A potential arbitrageur could borrow dollars and deposit euros to make a risk free profit. The following portfolio is an arbitrage portfolio. All values are recorded in dollars.

Portfolio	time t=0	time T= 6 months
1 short forward contract at forward price	0	$F(t,T) - X_T$
$1/(1+\alpha r_f)$ of foreign currency	$X_t/(1+\alpha r_f)$	X_T
$-X_t/(1+\alpha r_f)$ dollar currency	$-X_t/(1+\alpha r_f)$	$-X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)}$
Value	0	$F(t,T) - X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)}$

Plugging in values we find

$$F(t,T) - X_t \frac{(1+\alpha r_\$)}{(1+\alpha r_f)} = 0.0206$$

which is positive therefore this is an arbitrage portfolio.

c) The European commercial banks were undertaking the opposite of the trades we found in part (b). This imbalance in trades could have caused the arbitrage opportunity.

Question 3. A major currency pair is dollar-yen quoted in yen per dollar. Suppose the current price is \$1=¥82.10. Suppose also that the five-year dollar interest rate is 2.17% and the two-year rate is 0.78% (semi-annual, 30/360 daycount), and that the five-year and two-year yen rates are 0.63% and 0.41% respectively (semi-annual, act/365 daycount).

a) Calculate the forward price for dollar-yen five years forward. For simplicity use a 0.5 accrual factor, rather than 182/365 etc., for yen.

b) Suppose you were unable to trade forward contracts, but were able to trade spot FX and borrow or lend dollar and yen cash. How could you synthetically go long the forward contract in (a)?

c) Suppose you went long one forward contract from (a), and three years from now the FX rate and interest rates are unchanged. What is your profit or loss on the forward?

Solution. a) Using accrual factor $\alpha = 1/2$ the forward price is given by

$$F(t,T) = X_t \left(\frac{1 + r_\$/2}{1 + r_f/2}\right)^{2(T-t)}$$

where X_t is the price in dollars of one unit of foreign currency. In this case $X_0 = 1/82.10$. We find

$$F(0,5) = 0.0131482.$$

b) We want to build a portfolio which replicates holding a long forward contract.

Portfolio A	time t=0	time T= 5 years
1 long forward contract at forward price	0	$X_T - F(t,T)$
Value	0	$X_T - F(t,T)$

In portfolio B at time $t = 0$ we borrow $-X_t/(1 + r_f/2)^{10}$ dollars, trade dollars for yens spot, and deposit the yen cash. We deposit $1/(1 + r_f/2)^{10}$ yens at $t = 0$. At $T = 5$ years we trade yens for dollars spot and use this to repay our dollar loan. The transactions are shown in the following table.

Portfolio B	time t=0	time T= 5 years
$1/(1 + r_f/2)^{10}$ of foreign currency	$X_t/(1 + r_f/2)^{10}$	X_T
$-X_t/(1 + r_f/2)^{10}$ dollar currency	$-X_t/(1 + r_f/2)^{10}$	$-X_t\frac{(1+r_\$/2)^{10}}{(1+r_f/2)^{10}}$
Value	0	$X_T - X_t\frac{(1+r_\$/2)^{10}}{(1+r_f/2)^{10}}$

Since $F(0,5) = X_0\frac{(1+r_\$/2)^{10}}{(1+r_f/2)^{10}}$ we find that $V^A(5) = V^B(5)$ so these portfolios have the same value at any time $t \leq 5$ years.

c) We need to find $V_{F(0,5)}(3,5)$. In portfolio A, we have our current contract, which at maturity at year 5 gives 1 unit of foreign currency, and we pay $F(0,5)$ dollars in cash. In portfolio B, we have $Z_f(3,5)$ units of yen at $t = 3$ and borrow $-Z_\$(3,5)F(0,5)$ in cash. At time $T = 5$, we have 1 yen and owe $F(0,5)$ dollars in cash. So the values of A and B are the same at $T = 5$, i.e.,

$$V_A(5) = V_B(5) = 1 - F(0,5)$$

So their values of at time $t = 3$ must be the same by replication, and hence

$$\begin{aligned}
V_{F(0,5)}(3,5) &= V_A(3) = X_3 Z_f(3,5) - Z_\$(3,5)F(0,5) \\
&= X_3 Z_f(3,5) - Z_\$(3,5) X_0 \frac{Z_f(0,5)}{Z_\$(0,5)} \\
&= X_0 \left(Z_f(3,5) - Z_\$(3,5) \frac{Z_f(0,5)}{Z_\$(0,5)} \right) \\
&= 82.10 \left(\frac{1}{(1+0.78/200)^2} - \frac{1}{(1+0.41/200)^2} \frac{(1+2.17/200)^{10}}{(1+0.63/200)^{10}} \right) \\
&= -6.79
\end{aligned}$$

We know that at time $t = 3$, the forward price is $F(3,5) = X_3 Z_f(3,5)/Z_\$(3,5)$. Therefore, we could also have written

$$V_{F(0,5)}(3,5) = F(3,5)Z_\$(3,5) - Z_\$(3,5)F(0,5) = (F(3,5) - F(0,5))\, Z_\$(3,5)$$

which is identical to the formula we obtained for the *carry* of the trade with continuous compounding.

We should also check and see if the answer makes sense. So our asset has not gone up in value, so it makes sense that a long forward would go down in value. Therefore, at least the negative value makes sense.

Question 4. a) The one-year and two-year zero rates are 1% and 2% respectively. What is the one-year forward one-year rate (that is, f_{11}? Assume all rates are annually compounded. (This is a favorite question for interviewers.)

b) If the two-year forward one-year rate f_{21} is 3%, what is the three-year zero rate?

Solution. a) We solve

$$(1 + 1/100)(1 + f_{11}) = (1 + 2/100)^2$$

for f_{11}. Since $(1+1/100)(1+f_{11}) = 1.01 + 1.01 f_{11} \approx 1.01 + f_{11}$ and $(1+2/100)^2 \approx 1.04$, we get $f_{11} \approx 1.03$. Solving the equation exactly we find $f_{11} = 3.01\%$ so our approximation is very good.

b) We solve

$$(1 + 2/100)^2(1 + f_{21}) = (1 + r_3)^3$$

for r_3. We find $r_3 = 2.33\%$

Question 5. A bank has borrowing needs at time $T > 0$. Show that by combining an FRA trade today with a libor loan at time T, the bank can today lock in its interest cost for the period T to $T + \alpha$. Does the borrowing bank need to buy or sell the FRA to do this? What is the fixed rate that the bank locks in?

Solution. The bank should buy 1 FRA at the forward rate and receive a libor loan of 1 at time T. The payout of the FRA at time $T + \alpha$ is $\alpha(L_T[T, T + \alpha] - L_t[T, T + \alpha])$. The payout of the loan at time $T + \alpha$ is $-(1 + \alpha L_T[T, T + \alpha])$. Combining these 2 transactions the payout at time $T + \alpha$ will be $-(1 + \alpha L_t[T, T + \alpha])$. Thus if the bank buys 1 FRA at the forward libor rate and receives a libor loan of 1 at time T then it can lock in its interest cost for the period T to $T + \alpha$. The bank receives the fixed rate $L_t[T, T + \alpha] = \frac{Z(t,T) - Z(t,T+\alpha)}{\alpha Z(t,T+\alpha)}$.