

Math 210 – Problem Set 6 Solutions

Fall 2025

Question 1. Suppose $t \leq T_1 \leq T_2 \leq T_3$, where t is the current time, and $\Delta > 0$. Recall that $Z(T_1, T_2)$ is the price at time T_1 of a ZCB with maturity T_2 and $F(T_1, T_2, T_3)$ is the forward price at time T_1 for a forward contract with maturity T_2 on a ZCB with maturity T_3 .

- a) For each of the pairs of A and B in the table, choose the most appropriate relationship out of $\geq, \leq, =, ?$, where $?$ means the relationship is indeterminate. Give brief reasoning.

	A	$\geq, \leq, =, ?$	B
(i)	$Z(t, T_1)$		1
(ii)	$Z(T_1, T_1)$		1
(iii)	$Z(t, T_2)$		$Z(t, T_3)$
(iv)	$Z(T_1, T_2)$		$Z(T_1, T_3)$
(v)	$Z(T_1, T_3)$		$Z(T_2, T_3)$
(vi)	$Z(T_1, T_1 + \Delta)$		$Z(T_2, T_2 + \Delta)$
(vii)	$F(t, T_1, T_2)$		$F(t, T_1, T_3)$
(viii)	$F(t, T_1, T_3)$		$F(t, T_2, T_3)$
(ix)	$\lim_{T \rightarrow \infty} Z(t, T)$		0

Hint: Remember that at current time t , $F(t, \cdot, \cdot)$ is known but $Z(T, \cdot)$ is a random variable.

- b) What can you say about interest rates between T_1 and T_2 if

- i) $Z(t, T_1) = Z(t, T_2)$?
 ii) $Z(t, T_1) > 0$ and $Z(t, T_2) = 0$?

Solution.

A	Relation	B
$Z(t, T_1)$	$\leq$	1
$Z(T_1, T_1)$	$=$	1
$Z(t, T_2)$	$\geq$	$Z(t, T_3)$
$Z(T_1, T_2)$	$\geq$	$Z(T_1, T_3)$
$Z(T_1, T_3)$	$?$	$Z(T_2, T_3)$
$Z(T_1, T_1 + \Delta)$	$?$	$Z(T_2, T_2 + \Delta)$
$F(t, T_1, T_2)$	$\geq$	$F(t, T_1, T_3)$
$F(t, T_1, T_3)$	$\leq$	$F(t, T_2, T_3)$
$\lim_{T \rightarrow \infty} Z(t, T)$	$=$	0

Reasoning.

- (i) Receiving 1 in the future is worth less than receiving 1 now.
- (ii) At time T_1 , a ZCB maturing at T_1 pays 1 immediately, so value = 1.
- (iii) A payment at later date T_3 is worth less than one at earlier T_2 .
- (iv) Similarly, $Z(T_1, T_2) \geq Z(T_1, T_3)$.
- (v)–(vi) Comparison uncertain because future rates can fluctuate.
- (vii) Since $F(t, T_1, T_2) = \frac{Z(t, T_2)}{Z(t, T_1)}$ and $Z(t, T_2) \geq Z(t, T_3)$, we get $F(t, T_1, T_2) \geq F(t, T_1, T_3)$.
- (viii) Using $F(t, T_2, T_3) = \frac{Z(t, T_3)}{Z(t, T_2)} \geq \frac{Z(t, T_3)}{Z(t, T_1)} = F(t, T_1, T_3)$, hence inequality.
- (ix) Never receiving one dollar has zero value.

- Question 2.** a) A derivative contract pays $\alpha L_T[T, T + \alpha]$ at time $T + \alpha$. By constructing a portfolio of ZCBs and a libor deposit that replicates the payout, prove that the value at $t \leq T$ of the derivative contract is $Z(t, T) - Z(t, T + \alpha)$.
- b) Let $T_0, T_1, \dots, T_n$ be a sequence of times, with $T_{i+1} = T_i + \alpha$ for a constant $\alpha > 0$. Use your results from (a) to show that a floating leg of libor payments $\alpha L_{T_i}[T_i, T_i + \alpha]$ at times T_{i+1} , $i = 0, 1, \dots, n - 1$, has value at time $t \leq T_0$ equal to a simple linear combination of ZCB prices.
- c) Hence find the value of a spot-starting infinite stream of libor payments, that is, when $t = T_0 = 0$ and as $n \rightarrow \infty$.

Solution. Consider the portfolio:

Portfolio	time t	T	$T + \alpha$
1 ZCB with maturity T	$Z(t, T)$	1	$1 + \alpha L_T[T, T + \alpha]$
-1 ZCB with maturity $T + \alpha$	$-Z(t, T + \alpha)$	-1	
Value	$Z(t, T) - Z(t, T + \alpha)$		$\alpha L_T[T, T + \alpha]$

At time T we take the 1 from the ZCB which matures at T and place it in a Libor deposit. At time $T + \alpha$ we receive $1 + \alpha L_T[T, T + \alpha]$ from the deposit. Since this portfolio has the same payout at maturity as the derivative contract, by replication the value at t is $Z(t, T) - Z(t, T + \alpha)$.

- b) Receiving Libor payments $\alpha L_{T_i}[T_i, T_i + \alpha]$ at times $T_1, T_2, \dots, T_n$ has value

$$Z(t, T_0) - Z(t, T_1) + Z(t, T_1) - Z(t, T_2) + \dots - Z(t, T_n) = Z(t, T_0) - Z(t, T_n).$$

- c) If $t = T_0 = 0$ then the value is $Z(0, 0) - Z(0, T_n) = 1 - Z(0, T_n)$. Since $\lim_{n \rightarrow \infty} Z(0, T_n) = 0$, the value of a spot-starting infinite stream of Libor payments is 1.

Question 3. Let $T_0, T_1, \dots, T_n$ be a sequence of times with $T_{i+1} = T_i + \alpha$ for a constant $\alpha > 0$. A floating rate bond with notional 1, start date T_0 and maturity T_n pays libor coupons of $\alpha L_{T_i}[T_i, T_i + \alpha]$ at times T_{i+1} for $i = 0, 1, \dots, n-1$, and notional 1 at T_n .

- a) Find the price at $t < T_0$ of the floating rate bond.
- b) Using a replication argument, find the forward price at t for the floating rate bond (for a forward contract with maturity T) where $t < T < T_0$.

Solution. a) Find price at $t < T_0$:

$$P(t) = \sum_{i=0}^{n-1} \alpha L_t[T_i, T_i + \alpha] Z(t, T_{i+1}) + Z(t, T_n).$$

Substitute $L_t[T_i, T_i + \alpha] = \frac{Z(t, T_i) - Z(t, T_i + \alpha)}{\alpha Z(t, T_i + \alpha)}$:

$$P(t) = \sum_{i=0}^{n-1} (Z(t, T_i) - Z(t, T_{i+1})) + Z(t, T_n) = Z(t, T_0).$$

- b) For forward contract with maturity T , where underlying is a floating rate bond, define portfolios A and B:

Portfolio A: long forward contract with maturity T and delivery K . Value at time T is $V_K(T, T) = P(T) - K$, since at maturity we receive the floating rate bond, and pay the delivery price.

Portfolio B: long floating rate bond with value $P(t)$, short K ZCBs maturing at T . Value at time T is $P(T) - K$ and by replication we get $V_K(t, T) = P(t) - KZ(t, T)$.

Setting $V_K = 0$ yields the forward price

$$F(t, T) = \frac{P(t)}{Z(t, T)} = \frac{Z(t, T_0)}{Z(t, T)}.$$

Compare this formula with the forward on an asset paying no income. $F(t, T) = S_t e^{r(T-t)} = S_t / Z(t, T)$. So a forward contract on a floating rate bond has the same payout on a forward contract as an asset paying no income, where the asset is a ZCB with maturity T_0 .

Question 4. Consider a swap with fixed rate K and payment frequency α which starts at T_0 and ends at T_n .

- a) Express the value of the swap $V_K^{SW}(t)$ in terms of zero coupon bond prices only. The parameters K and α as well as the times $T_0, \dots, T_n$ should also appear in your final answer.

- b) Using your answer from part (a), show that for a given K , $V_K^{SW}(t)$ is bounded. That is, find finite ℓ and u independent of interest rates such that $\ell \leq V_K^{SW}(t) \leq u$. For $t = T_0 = 0$ (a spot-starting swap), $T_n = n$ and frequency $\alpha = 1$, find bounds in terms of n and K .
- c) Is the value of a forward contract on a stock S_t necessarily bounded above and below? Explain the key difference between the value of a swap and a stock forward contract. You may assume that the stock pays no income.

Solution. a) Value:

$$V_K^{SW}(t) = (y_t[T_0, T_n] - K)P_t[T_0, T_n] = Z(t, T_0) - Z(t, T_n) - K \sum_{i=1}^n \alpha Z(t, T_i).$$

using

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{\sum_{i=1}^n \alpha Z(t, T_i)},$$

and the formula for $P_t[T_0, T_n]$.

- b) Since $0 \leq Z(t, T_i) \leq 1$ and $Z(t, T_0) \geq Z(t, T_1) \geq \dots \geq Z(t, T_n)$,

$$\begin{aligned} V_K^{SW}(t) &= Z(t, T_0) - Z(t, T_n) - K \sum_{i=1}^n \alpha Z(t, T_i) \\ &\leq 1 - K \sum_{i=1}^n \alpha Z(t, T_i) \\ &\leq 1 - n\alpha K Z(t, T_n) \\ &\leq 1 \end{aligned}$$

So for the lower bound, we get

$$\begin{aligned} V_K^{SW}(t) &= Z(t, T_0) - Z(t, T_n) - K \sum_{i=1}^n \alpha Z(t, T_i) \\ &\geq 0 - 1 - K \sum_{i=1}^n \alpha Z(t, T_i) \\ &\geq 0 - 1 - n\alpha K Z(t, T_0) \\ &\geq 1 - n\alpha K \end{aligned}$$

For $\alpha = 1$, $1 - nK \leq V_K^{SW}(t) \leq 1$.

c) For forward on stock S_t with no income:

$$V_K(t, T) = (F(t, T) - K)e^{-r(T-t)} = S_t - Ke^{-r(T-t)}.$$

Assuming positive rates, $S_t - K \leq V_K(t, T) \leq S_t$. Since $S_t \geq 0$, we have $-K \leq V_K(t, T) \leq S_t$.

However, S_t is not bounded above in general, so the value of the forward contract is not necessarily bounded above.