Math 210, Fall 2025

Problem Set # 7

Question 1. Recall the proof that showed $\tilde{C}_K(t,T) = C_K(t,T)$ in class. Try using a similar proof to show that $\tilde{P}_K(t,T) = P_K(t,T)$. Explain why it fails.

Solution. As we did for $\tilde{C}_K = C_K$, assume $\epsilon = P_K(t,T) - \tilde{P}_K(t,T) > 0$. Consider a portfolio that consists of +1 American put - 1 Euro put. At time t, this has value $V(t) = 0$. You are short the Euro put, so the other party can choose to exercise at time T if $K > S_T$. If they exercise at time T, then you exercise too, and we get

$$V(T) = \epsilon + (K - S_T)^+ - (K - S_T)^+ \geq e > 0$$

This is an arbitrage portfolio. Proceed like we did in class for $\epsilon = \tilde{P}_K(t,T) - P_K(t,T) > 0$. The only case to consider is they exericise at time $t < T_0 < T$ (since we are short the American put, the other party might decide to do this). Then, we could sell our Euro put for $\tilde{P}_K(T_0,T)$ and receive payout $-(K - S_{T_0})$ at time T_0. This leaves us with

$$\begin{aligned} V(T_0) &= \epsilon - (K - S_{T_0}) + \tilde{P}_K(T_0,T) \geq \epsilon - (K - S_{T_0}) + (KZ(T_0,T) - S_{T_0}) \\ &\geq \epsilon + K(Z(T_0,T) - 1) \end{aligned}$$

If we have non-zero interest rates, then $Z(T_0,T) < 1$, and this does not give us a contradiction.

Question 2. Suppose the current one-year euro swap rate $y_0[0,1]$ is 1.74%, and the two-year and three-year swap rates are 2.24% and 2.55% respectively. Euro swap rates are quoted with annual payments and thus $\alpha = 1$.

a) Use bootstrapping to calculate $Z(0,1)$, $Z(0,2)$ and $Z(0,3)$ and obtain $P_0[0,3]$, the present value of a three-year annuity paying €1 per year.

b) Recall that the annually compounded zero rate for maturity T is the rate r such that

$$Z(0,T) = (1+r)^{-T}.$$

Calculate the one-year, two-year and three-year zero rates and compare them to the swap rates. For upward sloping yield curves, that is, when $y_0[0,T_2] > y_0[0,T_1]$ for $T_2 > T_1$, will zero rates be higher or lower than swap rates?

c) An approximate short-cut sometimes used on trading desks to calculate the present value of, say, a three-year annuity is to discount each payment at the three-year swap rate. In other words, it is assumed that

$$Z(0, n) = (1 + y_0[0, 3])^{-n}, \quad n = 1, 2, 3.$$

This is often called IRR (internal rate of return) discounting. Calculate the error in valuing the annuity in (a) this way.

d) Calculate the price of a three-year fixed rate bond of notional € 1 and annual coupons of 2.55% using the ZCB prices calculated in (a), and verify this equals the price obtained via IRR discounting at a rate of 2.55%.

e) The value of a swap with fixed rate K can be thought of as an annuity of amount $y_0[0, T_n] - K$ for period 0 to T_n. In yet another Wall Street quirk, euro swaps embedded in certain contracts are occasionally valued for cash settlement using IRR discounting at the current swap rate $y_0[0, T_n]$, rather than the correct valuation using ZCBs. The logic for this originally was to reduce disagreements between banks on cash settlement of swaps. The swap rate is easily observed and IRR discounting is then a deterministic formula, whilst the bootstrapping undertaken in (a) was deemed too complicated. Suppose all euro swaps suddenly moved to this type of valuation. Given your answers to parts a–c of this question, comment briefly on whether you would expect euro swap rates to rise, fall or stay the same.

Solution.

a) In class we found

$$y_t[T_0, T_n] = \frac{Z(t, T_0) - Z(t, T_n)}{P_t[T_0, T_n]} = \frac{Z(t, T_0) - Z(t, T_n)}{\sum_{i=1}^{n} \alpha Z(t, T_i)}.$$

Setting $t = T_0$ and $\alpha = 1$, we get the equations:

$$\begin{aligned} y_0[0, 1] &= \frac{1 - Z(0, 1)}{Z(0, 1)}, \\ y_0[0, 2] &= \frac{1 - Z(0, 2)}{Z(0, 1) + Z(0, 2)}, \\ y_0[0, 3] &= \frac{1 - Z(0, 3)}{Z(0, 1) + Z(0, 2) + Z(0, 3)}. \end{aligned}$$

Solving for the ZCB prices we find

$$\begin{aligned}
Z(0,1) &= \frac{1}{1+y_0[0,1]} = 0.982898, \\
Z(0,2) &= \frac{1-y_0[0,2]Z(0,1)}{1+y_0[0,2]} = 0.956556, \\
Z(0,3) &= \frac{1-y_0[0,3](Z(0,1)+Z(0,2))}{1+y_0[0,3]} = 0.926908.
\end{aligned}$$

Finally we have

$$P_0[0,3] = Z(0,1) + Z(0,2) + Z(0,3) = 2.86636,$$

the present value of a three-year annuity paying € 1 per year.

b) Let r_1, r_2, r_3 be the one-, two-, and three-year zero rates respectively. We have

$$Z(0,1) = \frac{1}{1+r_1}, \qquad Z(0,2) = \frac{1}{(1+r_2)^2}, \qquad Z(0,3) = \frac{1}{(1+r_3)^3}.$$

Solving these for the zero rates, we find

$$\begin{aligned}
r_1 &= Z(0,1)^{-1} - 1 = 1.7400\%, \\
r_2 &= Z(0,2)^{-1/2} - 1 = 2.2456\%, \\
r_3 &= Z(0,3)^{-1/3} - 1 = 2.5623\%.
\end{aligned}$$

From (a) and (b), we see that zero rates (in this case) are higher than swap rates for upward sloping yield curves. Looking at the expressions

$$\begin{aligned}
Z(0,2) = \frac{1}{(1+r_2)^2} &= \frac{1-y_0[0,2]Z(0,1)}{1+y_0[0,2]} \\
&= \frac{1+y_0[0,1]-y_0[0,2]}{(1+y_0[0,2])(1+y_0[0,1])} \\
&= \frac{1-\epsilon}{(1+y_0[0,2])(1+y_0[0,2]-\epsilon)}
\end{aligned}$$

where $\epsilon = y_0[0,2] - y_0[0,1]$. We want to find the dependence on ϵ. It is clear that the bounds on ϵ are

$$0 \leq \epsilon \leq y_0[0,2].$$

Manipulating the equation for $Z(0,2)$ we get

$$\frac{1}{(1+y_0[0,2])^2}\left(1+y_0[0,2] - \frac{y_0[0,2](1+y_0[0,2])}{1+y_0[0,2]-\epsilon}\right)$$

So as ϵ increases, it is clear that factor

$$\left(1 + y_0[0,2] - \frac{y_0[0,2](1 + y_0[0,2])}{1 + y_0[0,2] - \epsilon}\right)$$

is decreasing (just differentiate with respect to ϵ). So the factor must take its maximum value at $\epsilon = 0$ and its minimum at $\epsilon = y_0[0,2]$. Therefore, we get

$$(1 - y_0[0,2]^2)\frac{1}{(1 + y_0[0,2])^2} \leq \frac{1}{(1 + r_2)^2} \leq \frac{1}{(1 + y_0[0,2])^2}$$

This shows that $y_0[0,2] \leq r_2$. It also gives an upper bound that we will ignore. The general argument requires more algebra so we will ignore it.

c) Using the short-cut, we get

$$\begin{aligned} Z(0,1) &= (1 + y_0[0,3])^{-1} = 0.975134, \\ Z(0,2) &= (1 + y_0[0,3])^{-2} = 0.950886, \\ Z(0,3) &= (1 + y_0[0,3])^{-3} = 0.927242, \end{aligned}$$

which gives

$$Z(0,1) + Z(0,2) + Z(0,3) = 2.85326$$

for the annuity in (a).

d) Here $c = 2.55\%$. Using the ZCB prices from part (a) we find

$$P = cZ(0,1) + cZ(0,2) + cZ(0,3) + Z(0,3) = 1.$$

Using the IRR-discounted ZCB prices, we also get the same value.

e) In (a) we found the PV of a three-year annuity paying €1 per year with correct discounting to be 2.8664. In (c) we found the PV using IRR discounting to be 2.85326, which is slightly lower. If euro swaps suddenly moved to IRR valuation, the effective yield would have to drop so that price remains constant. Thus, euro swap rates would have to *fall*.

Question 3. Suppose euro swap rates are as given in the previous question. A hedge fund (HF) executes the following two trades with a dealer:

1. The HF pays fixed and receives floating on €100 million notional of a one-year swap at the forward swap rate.

2. The HF receives fixed and pays floating on €100 million notional of a three-year swap at the forward swap rate.

Assume bid-offer costs are negligible.

a) After one year, what net cashflow has the dealer paid to (or received from) the HF?

b) Suppose after one year, one-year and two-year euro swap rates are unchanged. What is the current value of the remaining part of the HF trade?

c) Suppose after one year, the one-year euro swap rate is unchanged but the two-year euro swap rate is now $Y\%$. What value of Y gives a total zero profit on the trade (at $T = 1$)?

d) Do you like the trades the HF executed? Discuss briefly the risks of the trade, in particular commenting on which interest rates the HF is exposed to.

Solution.

a) Let $N = 100 \times 10^6$. After one year, the net cashflow from the one-year swap is zero because $y_0[0,1] = L_0[0,1]$, thus

$$N(y_0[0,1] - L_0[0,1]) = 0.$$

The net cashflow from the three-year swap is

$$N(L_0[0,1] - y_0[0,3]) = -810{,}000.$$

Hence after one year the dealer has paid €810,000 to the HF.

b) The current value of the remaining part of the HF trade is the remaining value from the three-year swap:

$$\begin{aligned} N\big(y_0[0,3](Z(1,2) + Z(1,3)) - (Z(1,1) - Z(1,3))\big) &= N\big(y_0[0,3](Z(0,1) + Z(0,2)) - (1 - Z(0,2))\big) \\ &= 601{,}230.68. \end{aligned}$$

c) Now we have $y_1[1,3] = Y\%$, which means that

$$Z(1,3) = \frac{1 - YZ(0,1)}{1+Y}.$$

We need to solve

$$0 = 810{,}000 + N\big((1 - YZ(0,1))(y_0[0,3] + 1)/(1+Y) + y_0[0,3]Z(0,1) - 1\big)$$

for Y. We find

$$Y = 2.97\%.$$

d) The arbitrage-free one-year forward two-year swap rate is

$$y_0[1,3] = \frac{Z(0,1) - Z(0,3)}{Z(0,2) + Z(0,3)} = 2.97\%.$$

If in one year the two-year euro swap rate is below 2.97%, the HF will profit. If it is above, the HF will lose. Thus, the HF is effectively betting that two-year euro swap rates will remain below their arbitrage-free levels — unlikely over a full year.

At year 1, the value of the swap depends on $Z(1,2)$ and $Z(1,3)$, so the HF has exposure to both the one-year forward one-year swap rate and the one-year forward two-year swap rate.

Question 4. A fixed rate bond with notional 1 pays annual coupons of c at times $T_1, T_2, \ldots, T_n$ where $T_{i+1} = T_i + 1$ and notional 1 at time T_n.

a) Write down the bond price $B_c^{\text{FXD}}(t)$ at time $t \leq T_0$ in terms of ZCBs.

b) Suppose $t = T_0 = 0$. The yield of the bond is defined as the value Y such that

$$B_c^{\text{FXD}}(0) = \sum_{i=1}^{n} \frac{c}{(1+Y)^i} + \frac{1}{(1+Y)^n},$$

that is, the rate at which IRR discounting gives the bond price. By summing a geometric series, show that $B_c^{\text{FXD}}(0) = 1$ if and only if $Y = c$.

c) By writing a swap as the difference between a fixed rate bond and a floating rate bond, show that $B_c^{\text{FXD}}(0) = 1$ if and only if $c = y_0[0, T_n]$.

Remark. This exercise shows that the T-year spot swap rate is the bond coupon such that a T-maturity bond has price par (100% of notional).

Solution.

a) The price of the bond is obtained by discounting payments to today:

$$B_c^{\text{FXD}}(t) = c \sum_{i=1}^{n} Z(t, T_i) + Z(t, T_n).$$

b) Using the geometric series formula:

$$B_c^{\mathrm{FXD}}(0) = \frac{1}{(1+Y)^n} + \frac{c}{Y}\left(1 - \frac{1}{(1+Y)^n}\right) = \frac{c}{Y} + \frac{1}{(1+Y)^n}\left(1 - \frac{c}{Y}\right).$$

Thus $B_c^{\mathrm{FXD}}(0) = 1$ if and only if $Y = c$.

c) In class we showed that if $t = T_0 = 0$, then

$$V_c^{\mathrm{SW}}(0) = (y_0[0, T_n] - c)P_0[0, T_n] = 1 - B_c^{\mathrm{FXD}}(0).$$

Since $P_0[0, T_n] > 0$, we have $y_0[0, T_n] = c$ if and only if $B_c^{\mathrm{FXD}}(0) = 1$.

Question 2. Assume the continuously compounded interest rate has constant value 15%. The table below is for a futures contract maturing on day 5 with delivery price equal to the futures price. The underlying asset is a stock paying no income. The S_t column gives the stock price on each day. The $\Phi(t,T)$ column gives the futures price on each day. The MTM column lists the mark-to-market payments. The interest column lists the interest that will be accrued on the mark-to-market payment by the maturity date.

Fill in the table. Give at least four decimal places.

day	S_t	$\Phi(t,T)$	MTM	interest
0	2000			
1	1900			
2	2100			
3	2200			
4	2000			
5	2100			
		sum:		

Hint: Use Mathematica or a spreadsheet for the calculations.

Solution. Assuming the interest rates are constant we have $\Phi(t,T) = F(t,T) = S_t e^{r(T-t)}$ where $r = 15\%$. The MTM payments are $\Phi(t + i\Delta, T) - \Phi(t,T)$ where $\Delta = 1/365$. Each MTM payment accrues interest with rate $r = 15\%$. Note that the interest is expressed per annum. As an example we compute the payments on day 1:

$$\Phi(1/365, 5/365) = S_{1/365} e^{.15(5-1)/365} = 1903.1259$$

The MTM payment on day 1 is

$$\Phi(1/365, 5/365) - \Phi(0/365, 5/365) = 1903.1259 - 2004.1138 = -100.9880.$$

The interest accrued from the MTM payment by the maturity date is

$$-100.9880(e^{0.15(5-1)/365}-1)=-0.1661.$$

The remaining values are given in the table below.

day	S_t	$\Phi(t,T)$	MTM	interest
0	2000	2004.1138	0	
1	1900	1903.1259	-100.9880	-0.1661
2	2100	2102.5906	199.4648	0.2461
3	2200	2201.8090	99.2183	0.0816
4	2000	2000.8221	-200.9869	-0.0826
5	2100	2100.00	99.1779	0
		sum:	95.8660	0.0789