

Math 210, Fall 2025

Problem Set # 8

Question 1. For a stock paying dividends at continuous yield q with automatic reinvestment, show

$$\begin{aligned} S_t e^{-q(T-t)} - KZ(t, T) &\leq C_K(t, T) \leq S_t e^{-q(T-t)} \\ KZ(t, T) - S_t e^{-q(T-t)} &\leq P_K(t, T) \leq KZ(t, T) \\ S_t - K &\leq \tilde{C}_K(t, T) \leq S_t \\ K - S_t &\leq \tilde{P}_K(t, T) \leq K \end{aligned}$$

Solution. The thing to remember with stocks that return dividends that are immediately reinvested is that if you hold quantity A of the asset at time t , you end up holding $A \exp(q(T-t))$ of the asset at time T .

For the first inequality, hold $e^{-q(T-t)}$ of stock and borrow K ZCB. At maturity it has value $S_T - K \leq (S_T - K)^+ = C_K(T, T)$. Comparing the values at time t with the value of the option gives the lower bound. For the upper bound, just hold one asset, and certainly, $C_K(T, T) \leq S_T$. This gives the upper bound.

The second inequality with the Euro puts is similar.

For the American call, suppose you exercise at time t , then you get $(S_t - K)^+ \geq S_t - K$. So we must have $\tilde{C}_K(t, T) \geq S_t - K$. For the upper bound, suppose you exercise at time $t < t_0 < T$. Then, you obtain 1 asset and borrow cash K . This debt will grow to $KZ(t_0, T)$, and the value of the asset at time T will be $S_T \exp(q(T-t_0))$. Thus the total value of the portfolio is $S_T \exp(q(T-t_0)) - KZ(t_0, T)$. To get an upper bound, we can ignore the debt. Then, $S_T \exp(q(T-t_0)) - KZ(t_0, T) \leq S_T \exp(q(T-t))$. So by considering a portfolio that holds one unit of the asset, we get the upper bound.

The American put is similar.

A “ K call” is a call option with strike price K .

Question 2. A call butterfly with strikes (K_1, K_2, K_3) is a portfolio consisting of long 1 K_1 call, short 2 K_2 calls, and long 1 K_3 call, all with maturity T . Find the payout function at maturity T for a call butterfly with strikes $(70, 80, 110)$. Graph the payout at maturity as a function of the stock price.

Solution. The payout function at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq 70 \\ S_T - 70 & \text{if } 70 < S_T \leq 80 \\ S_T - 70 - 2(S_T - 80) = -S_T + 90 & \text{if } 80 < S_T \leq 110 \\ -S_T + 90 + S_T - 110 = -20 & \text{if } 110 < S_T. \end{cases}$$

Question 3. Assume all options are European style with maturity T . A “knockout” option has payout zero if the defined event occurs.

Consider the following eight options I-VIII, where $K_1 < K_2 < K_3$.

- I. K_1 call.
- II. K_1 call that knocks out (i.e., has payout zero) if $S_T > K_2$.
- III. K_1 call that knocks out if $S_t > K_2$ for any $0 \leq t \leq T$.
- IV. K_1 call that knocks out if $S_T < K_1$.
- V. K_1 call that knocks out if $S_t < K_1$ for any $0 \leq t \leq T$.
- VI. (K_1, K_2) call spread (long one K_1 call, short one K_2 call).
- VII. Digital call with strike K_1 and payout $K_2 - K_1$. In other words, the option whose payout at T is

$$\begin{cases} K_2 - K_1 & \text{if } S_T \geq K_1 \\ 0 & \text{if } S_T < K_1 \end{cases}$$

- VIII. (K_1, K_2, K_3) call ladder (long one K_1 call, short one K_2 call, short one K_3 call)

For each of the pairs of A and B in the table below, choose the most appropriate relationship between prices at time $t \leq T$ out of $=$, $\geq$, $\leq$, and $?$, where $?$ means the relationship is indeterminate.

Give justification for your answers.

Hint: Write down and compare the payouts at maturity for the options.

	A	=, ≥, ≤ or ?	B
(a)	I		VI
(b)	II		VI
(c)	II		III
(d)	I		IV
(e)	I		V
(f)	I		VII
(g)	VI		VII
(h)	VII		VIII
(i)	III		VIII
(j)	II		VII

Solution. First we write out the payouts of all of the options at maturity:

i) K_1 call:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T. \end{cases}$$

ii) K_1 call that knocks out if $S_T > K_2$:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T. \end{cases}$$

iii) K_1 call that knocks out if $S_t > K_2$ for any $0 \leq t \leq T$:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T < K_2 \quad \text{and } S_t < K_2 \\ 0 & \text{if } K_2 \leq S_T \quad \text{or } S_t > K_2. \end{cases}$$

iv) K_1 call that knocks out if $S_T < K_1$:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T. \end{cases}$$

v) K_1 call that knocks out if $S_t < K_1$ for any $0 \leq t \leq T$:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \quad \text{or } S_t < K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \quad \text{and } S_t > K_1. \end{cases}$$

vi) (K_1, K_2) call spread:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_1 - (S_T - K_2) = K_2 - K_1 & \text{if } K_2 < S_T \end{cases}$$

vii) Digital call with strike K_1 and payout $K_2 - K_1$.

$$g(S_T) = \begin{cases} 0 & \text{if } S_T < K_1 \\ K_2 - K_1 & \text{if } K_1 \leq S_T. \end{cases}$$

viii) (K_1, K_2, K_3) call ladder.

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_1 - (S_T - K_2) = K_2 - K_1 & \text{if } K_2 < S_T \leq K_3 \\ K_3 + K_2 - K_1 - S_T & \text{if } K_3 < S_T. \end{cases}$$

a) I and VI: If $S_T \leq K_2$ then we have I=VI. If $S_T > K_2$ then $K_2 - K_1 < S_T - K_1$ and so we have $I \geq VI$.

b) II and VI: If $S_T < K_2$ then we have II=VI. If $S_T > K_2$ then $0 \leq K_2 - K_1$ and so we have $II \leq VI$.

c) II and III: In all cases $II \geq III$.

d) I and IV: In this case $I = IV$.

e) I and V: In this case $I \geq V$.

f) I and VII: If $S_T < K_1$ then I=VII. However if $K_1 \leq S_T$ then they cannot be compared.

- g) VI and VII: If $S_T \leq K_1$ then VI=VII. If $K_1 < S_T \leq K_2$ then $S_T - K_1 \leq K_2 - K_1$ and if $K_2 \leq S_T$ then VI=VII. Thus we have VI $\leq$ VII.
- h) VII and VIII: If $S_T \leq K_3$ then VII $\geq$ VIII. If $K_3 \leq S_T$ then VII $\geq$ VIII.
- i) III and VIII: If $S_t > K_2$ but $K_3 < S_T$ then VIII $< 0 <$ III. However if $K_1 < S_T \leq K_2$ but $S_t > K_2$ then VIII $>$ III. Thus they cannot be compared.
- j) II and VII: In all cases II $\leq$ III.

	A	=, $\geq$, $\leq$ or ?	B
(a)	I	$\geq$	VI
(b)	II	$\leq$	VI
(c)	II	$\geq$	III
(d)	I	$=$	IV
(e)	I	$\geq$	V
(f)	I	?	VII
(g)	VI	$\leq$	VII
(h)	VII	$\geq$	VIII
(i)	III	?	VIII
(j)	II	$\leq$	VII

Question 4. Assume all options are European style with maturity T .

Consider the following ten options I-X, where $K_2 = K_1 + \beta$, $K_3 = K_2 + \beta$, and $\beta > 0$ is a constant.

I. K_1 call

II. K_1 put

III. K_3 , K_2 put spread, that is, +1 K_3 put and -1 K_2 put

IV. (K_1, K_2, K_3) call butterfly, that is, +1 K_1 call, -2 K_2 calls, and +1 K_3 call

V. K_1 call that knocks out if $S_T \geq K_2$

VI. Digital call with strike K_1 and payout β . In other words, the option whose payout at T is

$$\begin{cases} \beta & \text{if } S_T \geq K_1 \\ 0 & \text{if } S_T < K_1 \end{cases}$$

VII. (K_1, K_3) digital call spread with payout β , that is, a portfolio of +1 digital K_1 - call with payout β and -1 digital K_3 -call with payout β .

VIII. Digital put with strike K_3 and payout β that knocks out if $S_T \leq K_1$.

IX. A portfolio of +1 K_1 call and -2 K_2 calls, all of which knock out if $S_t < K_3$ for some t with $0 \leq t \leq T$.

X. A portfolio of +1 K_1 call and -2 K_2 calls, all of which knock out if $S_T > K_3$.

For each of the pairs of A and B in the table below, choose the most appropriate relationship between prices at time $t < T$ out of $=$, $\geq$, $\leq$, and $?$, where $?$ means the relationship is indeterminate.

	A	$=, \geq, \leq$, or $?$	B
(a)	I		II
(b)	II		III
(c)	III		IV
(d)	IV		V
(e)	IV		X
(f)	VII		VIII
(g)	III		VIII
(h)	IV		VII
(i)	II		IX
(j)	II		X

Hint for (f): Pay close attention to the payouts when $S_T = K_1$ and $S_T = K_3$. Do not assume the stock price is continuous.

Solution. As before the first step is to write out all of the payouts.

i) K_1 call:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T. \end{cases}$$

ii) K_1 put:

$$g(S_T) = \begin{cases} K_1 - S_T & \text{if } S_T \leq K_1 \\ 0 & \text{if } K_1 < S_T. \end{cases}$$

iii) K_3, K_2 put spread:

$$g(S_T) = \begin{cases} K_3 - S_T - (K_2 - S_T) = K_3 - K_2 = \beta & \text{if } S_T \leq K_2 \\ K_3 - S_T & \text{if } K_2 \leq S_T < K_3 \\ 0 & \text{if } K_3 \leq S_T. \end{cases}$$

Here we use $K_2 < K_3$.

iv) (K_1, K_2, K_3) call butterfly

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ S_T - K_1 - 2(S_T - K_2) = -S_T + K_1 + 2\beta = K_3 - S_T & \text{if } K_2 < S_T \leq K_3 \\ -S_T + K_3 + (S_T - K_3) = 0 & \text{if } K_3 < S_T. \end{cases}$$

v) K_1 call that knocks out if $S_T \geq K_2$:

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 \leq S_T < K_2 \\ 0 & \text{if } K_2 \leq S_T. \end{cases}$$

vi) Digital call with strike K_1 and payout β :

$$g(S_T) = \begin{cases} 0 & \text{if } S_T < K_1 \\ \beta & \text{if } K_1 \leq S_T. \end{cases}$$

vii) (K_1, K_3) digital call spread with payout β :

$$g(S_T) = \begin{cases} 0 & \text{if } S_T < K_1 \\ \beta & \text{if } K_1 \leq S_T < K_3 \\ 0 & \text{if } S_T \leq K_3. \end{cases}$$

viii) Digital put with strike K_3 and payout β that knocks out if $S_T \leq K_1$.

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ \beta & \text{if } K_1 < S_T \leq K_3 \\ 0 & \text{if } K_3 < S_T. \end{cases}$$

ix) Portfolio with 1 K_1 call and -2 K_2 calls all of which knock out if $S_t < K_3$.

$$g(S_T) = \begin{cases} 0 & \text{if } S_T < K_1 \text{ or } S_t \leq K_3 \\ S_T - K_1 & \text{if } K_1 < S_T < K_2 \text{ and } S_t \geq K_3 \\ -S_T - K_1 + 2K_2 = -S_T + 2\beta + K_1 & \text{if } K_2 < S_T \text{ and } S_t \geq K_3 \end{cases}$$

$$= \begin{cases} 0 & \text{if } S_T < K_1 \text{ or } S_t \leq K_3 \\ -S_T + 2\beta + K_1 = K_3 - S_T & \text{if } K_2 < S_T \text{ and } S_t \geq K_3. \end{cases}$$

x) A portfolio of 1 K_1 call and -2 K_2 calls all of which knock out if $S_T > K_3$.

$$g(S_T) = \begin{cases} 0 & \text{if } S_T \leq K_1 \\ S_T - K_1 & \text{if } K_1 < S_T \leq K_2 \\ -S_T + 2\beta + K_1 = K_3 - S_T & \text{if } K_2 < S_T \leq K_3 \\ 0 & \text{if } K_3 < S_T \end{cases}$$

- a) I and II: These are not comparable. In some situations $I \geq II$ and in other situations $I \leq II$.
- b) II and III: These are not comparable.
- c) III and IV: If $S_T \geq K_2$ then $III=IV$. If $S_T < K_2$ then $S_T - K_1 < \beta$ and so $III \geq IV$.
- d) IV and V: We find $IV \geq V$.
- e) IV and X: We find $IV = X$.
- f) VII and VIII: Note that if $S_T = K_1$ we have $VII \geq VIII$ but if $S_T = K_3$ we have $VIII \geq VII$ thus they are not comparable.
- g) III and VIII: If $K_2 \leq S_T < K_3$ then $K_3 - S_T \leq K_3 - K_2 = \beta$ and so $III \geq VIII$. However if $S_T \leq K_1$ then the payout of VIII is 0 but the payout of III is β and so $III > VIII$. Thus they are not comparable.
- h) IV and VII: If $K_1 < S_T \leq K_2$ then $S_T - K_1 \leq K_2 - K_1 \leq \beta$. If $K_2 < S_T \leq K_3$ then $K_3 - S_T \leq K_3 - K_2 \leq \beta$. Thus $IV \leq VII$.
- i) II and IX: If $K_2 < S_T$ then $K_3 - S_T \leq K_3 - K_2 = \beta$ and so $II \geq IX$.

j) II and X: They are not comparable.

	A	$=, \geq, \leq$, or ?	B
(a)	I	?	II
(b)	II	?	III
(c)	III	$\geq$	IV
(d)	IV	$\geq$	V
(e)	IV	$=$	X
(f)	VII	?	VIII
(g)	III	?	VIII
(h)	IV	$\leq$	VII
(i)	II	$\geq$	IX
(j)	II	?	X