Math 210, Fall 2025

Problem Set # 9

Question 1. Assume all options are European style with maturity T. You may assume the underlying asset is a stock paying no income.

a) Recall that a call butterfly with strikes $(K, K+\beta, K+2\beta)$, for some fixed $\beta > 0$, is a portfolio consisting of $+1$ K call, $+1$ $(K+2\beta)$ call, and -2 $(K+\beta)$ calls, all with maturity T. Using put-call parity (or otherwise), restate the call butterfly as a portfolio consisting solely of puts.

b) A call condor is a portfolio consisting of $+1$ K call, -1 $(K+\beta)$ call, -1 $(K+2\beta)$ call, and $+1$ $(K+3\beta)$ call, all with maturity T. Draw the payout of the call condor, and express the condor as a portfolio consisting solely of call butterflies.

c) A call ladder consists of $+1$ K call, -1 $(K+\beta)$ call and -1 $(K+2\beta)$ call, all with maturity T. What relationships hold between the prices at time $t \leq T$ of the call ladder, butterfly and condor with common maturity T?

Solution. a) Put-call parity states

$$C_K(t,T) = V_K(t,T) + P_K(t,T)$$

where $V_K(t,T)$ is the value of a forward contract with delivery price K. A call butterfly is the portfolio $H(t)$ given by

$$H(t) = C_K(t,T) - 2C_{K+\beta}(t,T) + C_{K+2\beta}(t,T).$$

Substituting in put-call parity we get

$$\begin{aligned} H(t) &= V_K(t,T) + P_K(t,T) - 2\left(V_{K+\beta}(t,T) + P_{K+\beta}(t,T)\right) + V_{K+2\beta}(t,T) + P_{K+2\beta}(t,T) \\ &= P_K(t,T) - 2P_{K+\beta}(t,T) + P_{K+2\beta}(t,T) + V_K(t,T) - 2V_{K+\beta}(t,T) + V_{K+2\beta}(t,T). \end{aligned}$$

Recall the value of a forward contract on an asset satisfies

$$V_K(t,T) = (F(t,T) - K)Z(t,T)$$

therefore

$$\begin{aligned} &V_K(t,T) - 2V_{K+\beta}(t,T) + V_{K+2\beta}(t,T) \\ &= (F(t,T) - K)Z(t,T) - 2(F(t,T) - K - \beta)Z(t,T) + (F(t,T) - K - 2\beta)Z(t,T) \\ &= Z(t,T)(-K + 2K + 2\beta - K - 2\beta) \\ &= 0 \end{aligned}$$

therefore

$$H(t) = P_K(t,T) - 2P_{K+\beta}(t,T) + P_{K+2\beta}(t,T).$$

This problem can also be solved by computing the payout of the call butterfly at maturity and finding a portfolio of puts with the same payout at maturity.

b) The payout of the call condor at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K \\ S_T - K & \text{if} \quad K < S_T \leq K+\beta \\ S_T - K - (S_T - K - \beta) = \beta & \text{if} \quad K+\beta < S_T \leq K+2\beta \\ \beta - S_T + K + 2\beta = -S_T + K + 3\beta & \text{if} \quad K+2\beta < S_T \leq K+3\beta \\ -S_T + K + 3\beta + S_T - K - 3\beta = 0 & \text{if} \quad K+3\beta < S_T. \end{cases}$$

The payout of the call butterfly at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K \\ S_T - K & \text{if} \quad K < S_T \leq K+\beta \\ S_T - K - 2(S_T - K - \beta) = -S_T + K + 2\beta & \text{if} \quad K+\beta < S_T \leq K+2\beta \\ -S_T + K + 2\beta + S_T - K - 2\beta = 0 & \text{if} \quad K+2\beta < S_T. \end{cases}$$

Consider the portfolio consisting of long one $(K, K+\beta, K+2\beta)$ and long one $(K+\beta, K+2\beta, K+3\beta)$ call butterfly. Its payout at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \leq K \\ S_T - K & \text{if} \quad K < S_T \leq K+\beta \\ -S_T + K + 2\beta + (S_T - K - \beta) = \beta & \text{if} \quad K+\beta < S_T \leq K+2\beta \\ -S_T + K + 3\beta & \text{if} \quad K+2\beta < S_T \leq K+3\beta \\ 0 & \text{if} \quad K+3\beta < S_T. \end{cases}$$

Therefore the portfolio consisting of long one $(K, K+\beta, K+2\beta)$ and long one $(K+\beta, K+2\beta, K+3\beta)$ call butterfly is equivalent to the portfolio consisting of one call condor.

c) The payout of the call ladder at maturity is

$$g(S_T) = \begin{cases} 0 & \text{if} \quad S_T \le K \\ S_T - K & \text{if} \quad K < S_T \le K + \beta \\ S_T - K - (S_T - K - \beta) = \beta & \text{if} \quad K + \beta < S_T \le K + 2\beta \\ \beta - S_T + K + 2\beta = -S_T + K + 3\beta & \text{if} \quad K + 2\beta < S_T. \end{cases}$$

Let $L_K(t,T)$ denote the price of a call ladder, $B_K(t,T)$ the price of a call butterfly, and $D_K(t,T)$ the price of a call condor. By direct comparison of the payouts at maturity we find

$$L_K(t,T) \le D_K(t,T)$$
$$L_K(t,T) \; ? \; B_K(t,T)$$
$$B_K(t,T) \le D_K(t,T)$$

Question 2. a) Using put-call parity and bounds on a European call, or otherwise, show that the price of a European put on a stock paying no dividends satisfies

$$\max\{0, KZ(t,T) - S_t\} \le P_K(t,T) \le KZ(t,T).$$

b) Show that the price of an American put on the non-dividend paying stock satisfies

$$\max\{0, K - S_t\} \le \tilde{P_K}(t,T) \le K.$$

c) Give an example to show that the American put can be worth more than the European put, that is, where immediate exercise of the American gives payout at times T strictly greater than the payout of the European. **Hint:** Exercising early means the strike price K will be received early. Consider under which environments this will be preferable.

Solution. a) In class we showed

$$\max\{0, S_t - KZ(t,T)\} \le C_K(t,T) \le S_t.$$

By put-call parity we have

$$\begin{aligned} C_K(t,T) &= V_K(t,T) + P_K(t,T) \\ &= (F(t,T) - K)Z(t,T) + P_K(t,T) \\ &= (S_t/Z(t,T) - K)Z(t,T) + P_K(t,T) \\ &= S_t - KZ(t,T) + P_K(t,T). \end{aligned}$$

Therefore

$$S_t \geq C_K(t,T) = S_t - KZ(t,T) + P_K(t,T)$$

and rearranging we find

$$P_K(t,T) \leq KZ(t,T).$$

For the lower bound there are two cases. If $S_t - KZ(t,T) \leq 0$ then

$$0 \leq C_K(t,T) = S_t - KZ(t,T) + P_K(t,T)$$

and we find

$$KZ(t,T) - S_t \leq P_K(t,T).$$

In the other case $S_t - KZ(t,T) \geq 0$ and we find

$$S_t - KZ(t,T) \leq C_K(t,T) = S_t - KZ(t,T) + P_K(t,T).$$

Rearranging we get

$$0 \leq P_K(t,T).$$

Both cases together imply

$$\max\{0, KZ(t,T) - S_t\} \leq P_K(t,T).$$

b) If the American put is never exercised we receive nothing which implies $\tilde{P_K}(t,T) \geq 0$. If the American put is exercised before T we receive the strike and sell the stock (both at time t) thus $\tilde{P_K}(t,T) \geq K - S_t$. These two inequalities imply

$$\max\{0, K - S_t\} \leq \tilde{P_K}(t,T).$$

For the upper bound note that $\tilde{P_K}(T,T) \leq K$ and so by monotonicity theorem we must have $\tilde{P}_K(t,T) \leq K$.

c) Suppose a stock has price $S_0 = 5$ and the one-year annually compounded interest rate is $r = 50\%$.

Consider a 20-strike European put. The payout at maturity is $P_K(1,1) = (K - S_T)^+ \leq K = 20$.

Consider a K-strike American put. If we exercise the American put immediately we receive $K - S_0 = 20 - 5 = 15$ which we can deposit in a bank account. After one year's time we will have an amount $(K - S_0)(1 + r) = 15(1 + .5) = 22.5$.

We find $\tilde{P}_K(1,1) > P_K(1,1)$ and thus $\tilde{P}_K(0,1) > P_K(0,1)$.

Question 3. a) By considering a portfolio of ZCBs and a call option, prove that the price at time $t \le T$ of a call option with strike K on a stock that pays no dividends satisfies

$$C_K(t,T) \ge \max\{S_t - KZ(t,T), 0\}.$$

b) Hence show that if $t \le T_1 \le T_2$,

$$C_K(t,T_2) \ge C_k(t,T_1).$$

c) Does the same result hold for puts? That is, prove or find a counter example to the statement

$$P_K(t,T_2) \ge P_K(t,T_1) \qquad \text{for} \qquad t \le T_1 \le T_2.$$

The same result does not hold for puts.

Solution. a) Consider the following two portfolios:

Portfolio A	time t	T
long 1 call	$C_K(t,T)$	$(S_T - K)^+$
K ZCBs with maturity T	$KZ(t,T)$	K
Value	$C_K(t,T) + KZ(t,T)$	$(S_T - K)^+ + K$

Portfolio B	time t	T
long 1 share stock	S_t	S_T
Value	S_t	S_T

If $S_T \ge K$ then $V_A(T) = S_T - K + K = S_T = V_B(T)$. If $S_T \le K$ then $V_A(T) = K \ge S_T = V_B(T)$. Thus in all situations we have $V_A(T) \ge V_B(T)$. By the monotonicity theorem we must have $V_A(t) \ge V_B(t)$. This implies

$$C_K(t,T) + KZ(t,T) \ge S_t$$
$$C_K(t,T) \ge S_t - KZ(t,T).$$

Since the payout of a call is non-negative we always have $C_K(t,T) \ge 0$. We conclude

$$C_K(t,T) \ge \max\{S_t - KZ(t,T), 0\}.$$

b) There are 2 cases. If $S_{T_1} \leq K$ then the first put is never exercised and so $C_K(t, T_1) = 0$ which implies $C_K(t, T_2) \geq C_K(t, T_1)$ since $C_K(t, T_2) \geq 0$ always.

In the second case $S_{T_1} \geq K$ and so the first put is exercised. When we exercise the put we receive 1 share of stock and pay K. we have borrowed an amount K at time T_1 and at time T_2 it will grow to an amount $K/Z(T_1, T_2) \geq K$ since $Z(T_1, T_2) \leq 1$. These transactions are represented in Portfolio B below.

Portfolio B	time t	T_1	T_2
long 1 call with maturity T_1	$C_K(t, T_1)$	$(S_{T_1} - K)^+$	0
long 1 share stock	0	S_{T_1}	S_{T_2}
cash	0	$-K$	$-K/Z(T_1, T_2)$
Value	$C_K(t, T_1)$	?	$S_{T_2} - K/Z(T_1, T_2)$

Since $K/Z(T_1, T_2) \geq K$ (as noted above) we have

$$V_B(t, T) = S_{T_2} - K/Z(T_1, T_2) \leq S_{T_2} - K.$$

Now consider Portfolio B.

Portfolio A	time t	T_2
long 1 call with maturity T_2	$C_K(t, T_2)$	$(S_{T_2} - K)^+$
Value	$C_K(t, T_2)$	$(S_{T_2} - K)^+$

Since $(S_{T_2} - K)^+ \geq S_{T_2} - K$ we have $V_A(1) \geq V_B(1)$ which implies $C_K(t, T_1) \leq C_K(t, T_2)$.

c) The same result does not hold for puts. Suppose $S_0 = 2$, $Z(0, 1) = 0.95$ and $Z(0, 2) = 0.9$. Consider a 100-strike one-year European put $P_{100}(0, 1)$ and a 100-strike two-year European put $P_{100}(0, 2)$. From the inequalities in question 2 we have

$$P_{100}(0, 2) \leq 100Z(0, 2) = 90$$

and

$$P_{100}(0, 1) \geq KZ(0, 1) - S_0 = 95 - 2 = 93.$$

This shows $P_{100}(0, 1) > P_{100}(0, 2)$.

Question 4. A stock that pays no dividends has price today of 100. In one year's time the stock is worth 110 with probability 0.75 and 85 with probability 0.25. The one-year annually compounded interest rate is 5%.

a) Calculate the forward price of the stock for a forward contract with maturity one year.

b) Calculate the price of a one-year European put option with strike 100.

c) Suppose you observe that the put option in part (b) has a market price of 4. Determine an arbitrage portfolio and calculate how much profit is generated at time $T = 1$ by this portfolio.

Solution. a)

$$F(0,1) = S_0(1+r)^1 = 100(1+.05)^1 = 105.$$

b) We have $u = 10\%$ and $d = -15\%$ thus

$$p^* = \frac{r-d}{u-d} = \frac{.05+.15}{.1+.15} = \frac{2}{2.5} = \frac{4}{5}.$$

We find

$$P_100(0,1) = Z(0,1)E_*[(100 - S_T)^+] = \frac{1}{1.05} \cdot \frac{1}{5} \cdot (100 - 85) = 2.8571.$$

c) The market price of the put is high therefore you should short the put. Before we can do that we need to find a portfolio of ZCBs and stock which replicates the payout of the put. Suppose the portfolio has λ stocks and μ ZCBs with maturity 1. Then the portfolio should satisfy

$$110\lambda + \mu = 0$$
$$85\lambda + \mu = 15$$

thus $\lambda = -3/5$ and $\mu = 66$. The value of this portfolio at time 0 is

$$-\frac{3}{5} \cdot 100 + \frac{66}{1+.05} = \frac{20}{7},$$

which we recognize as the fair price of the put.

The following portfolio is an arbitrage portfolio:

Portfolio A	time t	T
short 1 put	$-P_{100}(0,1) = -4$	$-(100 - S_T)^+$
$-3/5$ shares stock	$-3 \cdot 100/5$	$(-3/5)S_1$
66 ZCBs with maturity 1	$66/1.05 = 62.8571$	66
1.14 ZCBs with maturity 1	1.14	$1.14(1.05) = 1.97$
Value	0	1.97

By replication we have

$$-(100 - S_1)^+ - 3/5S_1 + 66 = 0$$

and so the value of the portfolio will be 1.97 in all cases. With this portfolio you generate a profit of 1.97 at time $T = 1$.