Lec 10 (3.1)

Ex 2.5.13 Complex number example.

Determinants: Proved that $\underset{n\times n}{A}$ is invertible iff

rank$(A) = n$.

We want to define a new notion of invertibility

det(A) that's a number st A is invertible iff

$$\det(A) \neq 0$$

Case $n=1$ $A = [a_{11}]_{1\times 1}$ is full rank iff

$a_{11} \neq 0$. So simply set $\det(A) = a_{11}$.

Case: $n=2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \sim \begin{bmatrix} 1 & a_{12}/a_{11} \\ a_{21} & a_{22} \end{bmatrix}$$

$(a_{11} \neq 0)$

$$\sim \begin{bmatrix} 1 & a_{12}/a_{11} \\ 0 & a_{22} - \frac{a_{12}a_{21}}{a_{11}} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & a_{12}/a_{11} \\ 0 & 1 \end{bmatrix}$$ if $a_{11}a_{22} - a_{12}a_{21} \neq 0$

Because if it were then we could not divide the 2nd row by it.

So $\text{rank}(A) = 2$ if $a_{11} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$

Similarly by exchanging the 1st 2 rows:

If $a_{21} \neq 0$ and $a_{11}a_{22} - a_{12}a_{21} \neq 0$

then $\text{rank}(A) = 2$

If both $a_{11} = 0$ and $a_{21} = 0$ then $a_{11}a_{22} - a_{12}a_{21} = 0$

So $a_{11}a_{22} - a_{12}a_{21} \neq 0 \Rightarrow$ at least one of a_{11} or a_{21} $\neq 0$. So det

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Case $n = 3$: reduce A to row echelon form and obtain:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

and we can show that $\text{rank}(A) = 3 \Rightarrow \det(A) \neq 0$

Let us generalize this idea: a general term is of the form

$a_{1\cdot}\; a_{2\cdot}\; a_{3\cdot}$ where the dots consist of #s 1, 2, 3

Notice also that ALL 3 numbers 1, 2 and 3 appear in each term.

$$a_{12}\, a_{23}\, a_{31}.$$

So lets collect the possibilities for the dots: $(1,2,3)$, $(2,3,1)$, $(3,1,2)$, $(1,3,2)$, $(2,1,3)$, $(3,2,1)$ 6 POSSIBILITIES

$6 = 3! = \#$ of perm. of 3 letters.

Permutations

$\{1,2,\dots,n\}$ $\sigma = (2,3,1,5,\dots)$ Arrangement in a specific order.

$S_n = \{$ set of all permutations of $1,\dots,n\}$ $|S_n| = n! \approx n$

$= (n$ choices for first letter$)(n-1$ for the next$) \dots = n!$

Inversion: $(1,3,2) \in S_3$. Count the # of pairs such that (i,j) are not in order Ex: $(3,2)$. There are $\binom{3}{2} = \frac{3!}{2!\,1!} = 3$ pairs i,j and only 1 is out of order

Ex: $(1,3,2,4,5) \in S_5$ has 1 inversion

how many possible: $\binom{5}{2} = \frac{5\cdot 4^2}{1\cdot 2} = 10$. Here is one that achieves it. $(5,4,3,2,1) = 4+3+2+1 = 10$

Def: Given $\sigma = (p_1, p_2, \dots p_n) \in S_n$ Define $N(\sigma) = \#$ of inversions

We say σ is

ODD: if $N(\sigma)$ is odd EVEN $N(\sigma)$ is even.

Parity $P(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is ODD} \\ -1 & \text{if } \sigma \text{ is EVEN} \end{cases}$

Ex: $(1,3,2,4,5)$ $N(\sigma) = 1$. Let us swap $(2,5)$ Then we get $(1,3,5,4,2) = \sigma'$ $N(\sigma') = 1+2+1 = 4$

Notice that σ is ODD but σ' is EVEN.

Theorem If elements in a permutation ~~are swapped~~ then the permutation switches PARITY.

Let's recall $\det(A) = \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$

$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$

Notice that The 1st three permutations are $(1\,2\,3)$ $(2\,3\,1)$ $(3\,1\,2)$ have $N = 0, 2, 2$ so $P = +++$ respectively

Similarly $(1\,3\,2)$ $(2\,1\,3)$ $(3\,2\,1)$ have $N = 1, 1, 3$ "

For $n=3$ one way to remember signs is write

3 negative terms

$$\left|\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix}\right|$$

3 +ve terms

$\det(A) = |A|$ (sometimes)

Ex: $|-6| = -6$ $\quad \begin{vmatrix} -4 & 6 \\ -2 & 5 \end{vmatrix} = -20 - (-2 \cdot 6)$

$= -20 + 12 = -8$

$$\begin{vmatrix} 2 & -5 & 2 \\ 6 & 1 & 0 \\ -3 & -1 & 4 \end{vmatrix} = \begin{matrix} 2 & -5 & 2 & 2 & -5 \\ 6 & 1 & 0 & 6 & 1 \\ -3 & -1 & 4 & -3 & -1 \end{matrix} = 8 + 0 - 12 + 6 + 0 + 120 = 122$$

$$|S_4| = 4! = 4\cdot 3\cdot 2 = 24$$

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{vmatrix} 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & 6 \\ -2 & 1 & 0 & 0 \\ -7 & 3 & 0 & 0 \end{vmatrix}$$

(1 2 3 4)
(2 3 4 1)
(3 4 1 2)
4 1 2 3
⋮
too many

In rows 1 and 2 have two nonzero options: 3 and 4

In rows 3 and 4, only columns 1 and 2

3412	4312	$\Rightarrow$	$2+2$	$3+2$	$\Rightarrow$	$+$	$-$
3421	4321		$2+2+1$	$3+2+1$		$-$	$+$

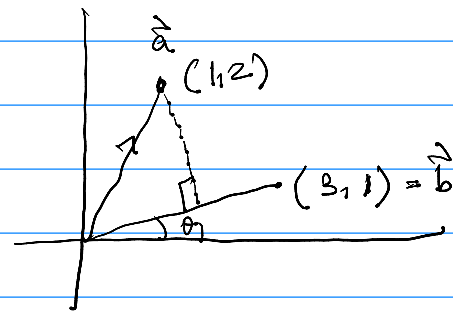
$$\det(A) = [-2\cdot 6\cdot(-2)(3)] - [3\cdot 1\ (-2)\ 3] - [(-2)6\ 1(-7)]$$
$$+ [3\cdot 1\ 1(-7)] = 72 + 18 - 84 - 21 = 90 - 105 = -15$$

Find all x satisfying
$$0 = \begin{vmatrix} x^2 & x & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{vmatrix} = \begin{matrix} x^2 & x & 1 & x^2 & x \\ 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 4 & 2 \end{matrix} = \begin{matrix} x^2 + 4x + 2 \\ -4 - 2x^2 - x \end{matrix}$$
$$= -x^2 + 3x - 2 = 0 \qquad x^2 - 3x + 2 = 0$$
$$(x-1)(x-2) = 0$$

Geometric Interpretation

$\vec{a}$ $\vec{b}$ are vectors we can find their lengths.

$\vec{a} = (1,2)$ $\quad \|\vec{a}\| = \sqrt{1^2+2^2}$



$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

$$= (1,2)\cdot(3,1) = 3+2 = 5$$

$\|\vec{a}\| = \sqrt{5}\,\sqrt{10} = \sqrt{2}\cdot 5$ $\qquad \cos\theta = \vec{a}$

Why is $(a_1b_1 + a_2b_2) = \|a\|\|b\|\cos\theta$? $\cos\theta_1 = \frac{b_1}{\|\vec{b}\|}$ $\qquad \cos\theta_2 = \frac{a_1}{\|\vec{a}\|}$

$$\cos(\theta_2-\theta_1) = \cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1$$
$$= \frac{a_1b_1}{\|\vec{a}\|\|\vec{b}\|} + \frac{a_2b_2}{\|\vec{a}\|\|\vec{b}\|}$$

It follows from the angle formulas. So how to prove the angle formulas? They can be found on wikipedia but the easiest is using Euler's identity.

$e^{i\alpha} = \cos\alpha + i\sin\alpha$ $\qquad e^{i\alpha}e^{-i\beta} = e^{i(\alpha-\beta)} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$

$$(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$

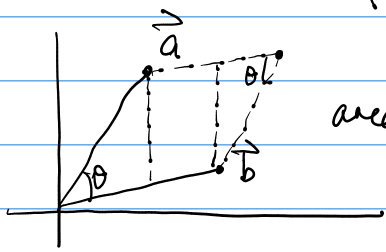
$\Rightarrow \cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha-\beta)$

Similarly $a\times b = \|a\|\,\|b\|\sin\theta\,\hat{n}$ where $\hat{n}$ is perpendicular to the plane of a and b.

We can write $\vec{a} \times \vec{b}$ using determinants. $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= \hat{i}(a_2 b_3 - a_3 b_2) + \hat{j}(a_3 b_1 - b_2 a_3) + \hat{k}(a_1 b_3 - a_3 b_1)$

Theorem: The area of a parallelogram determined by $\vec{a} = (a_1, a_2)$ $\vec{b} = (b_1, b_2)$ is $\left| \det\left(\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \right|$



area $= \frac{1}{2}\sqrt{\|a\| \cos\theta} \|a\| \sin\theta \times 2$

$+ (\|b\| - \|a\| \cos\theta)\|a\| \sin\theta$

$= \|b\| \|a\| \sin\theta$

$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$

$= a_2 b_1 - a_1 b_2$

So we get $\|a\| \|b\| \sin\theta$

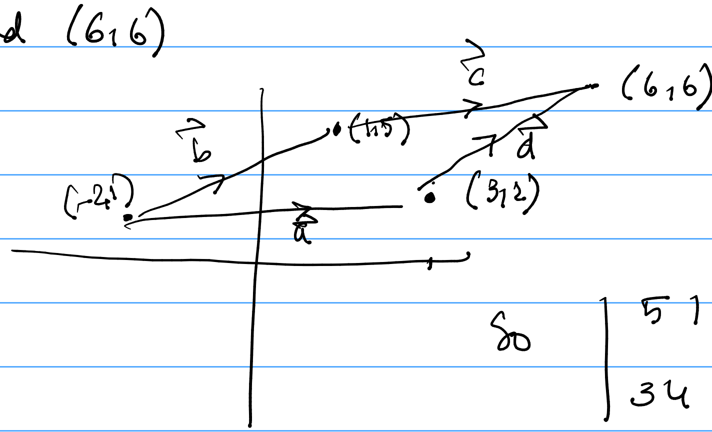
$= |a_2 b_1 - a_1 b_2|$

$= \left| \det\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \right|$

Theorem: Volume of a parallelepiped in $\mathbb{R}^3$ is (determined by $\vec{a}$ $\vec{b}$ $\vec{c}$) is

$$\left| \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \right|$$

Ex: Find area of $\square$ with vertices $(-2,1)$ $(1,5)$ $(3,2)$ and $(6,6)$



$\vec{a} = (5,1)$ $\vec{b} = (3,4)$

$\vec{c} = (5,1)$ ✓ $\vec{d} = (3,4)$ ✓

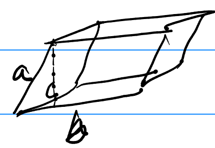
So $\begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix} = 20 - 3 = 17$

Ex: Do $(-4, 1, 0)$ $(2, 3, 1)$ $(-8, 9, 2)$ lie on the same plane?

$$\det(A) = \begin{vmatrix} -4 & 1 & 0 \\ 2 & 3 & 1 \\ -8 & 9 & 2 \end{vmatrix} = \begin{matrix} -4 & 1 & 0 & -4 & 1 \\ 2 & 3 & 1 & 2 & 3 \\ -8 & 9 & 2 & -8 & 9 \end{matrix} = -24 - 8 + 0 - 0 + 36 - 4 = 0$$

$\Rightarrow$ volume of  $= 0$ so they line on the same plane.

Proof of parallelepiped theorem:



area of base $\times$ perpendicular height

Area of base $= \|c \times b\|$ Height $= |a \cdot n|$

$c \times b = \hat{n} \|c\| \|b\| \sin\theta$ $= |\|a\| \|n\| \cos\varphi| = \|a\| |\cos\varphi|$

$\|c \times b\| = \|c\| \|b\| |\sin\theta|$

So volume $= \| a \cdot (c \times b) \| = \|a\| \|c \times b\| |\cos\varphi| = \|a\| |\cos\varphi| \, \|c\| \|b\| |\sin\theta|$

$$c \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

If $a = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$

$$\Rightarrow a \cdot (c \times b) = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$