lec 25

Concepts:

1) Solving a system by turning it into an n^{th} order equation

2) Writing general solutions for systems with non defective matrices

3) Solving initial value problems.

Ex: ① $x_1'(t) = x_1 + 2x_2$ | $x_1(0) = 1$ $x_1(t)$, $x_2(t)$

② $x_2'(t) = 2x_1 - 2x_2$ | $x_2(0) = 0$ ⟵ Initial value problem.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \end{pmatrix}$$

$x'(t) = Ax(t)$ $x(0) = \vec{v}$

$A_{n\times n}$ $x(t)_{n\times 1}$

Method 1:

$$D^2 x_1 = Dx_1 + 2Dx_2 = Dx_1 + 4x_1 - 4x_2$$

$$= Dx_1 (-2Dx_1 + 2Dx_1) + 4x_1 - 4x_2$$

$$= -Dx_1 + 6x_1 \;]\text{ eliminated } x_2$$

Manipulate the equations to eliminate $x_2(t)$.

"n^{th} order linear ODE is $x_1(t)$" (2nd order here)

$D(Dx_1 = x_1 + 2x_2)$

$$(D^2 + D - 6)x_1 = 0 \qquad (D+3)(D-2)x_1 = 0$$

$$x_1 = Ae^{-3t} + Be^{2t}$$

$m_{-3} = 1$ $m_2 = 1$

Using ★1

I know x_1

$$2x_2 = Dx_1 - x_1 \Rightarrow x_2 = \frac{1}{2}(Dx_1 - x_1)$$

$$= \frac{1}{2}\left(\underbrace{-3Ae^{-3t} + 2Be^{2t}} - Ae^{-3t} - Be^{2t}\right)$$

$$= -2Ae^{-3t} + \frac{B}{2}e^{2t}$$

Higher order systems can be written as 1st order systems

Write

2nd derivative

$$\boxed{\begin{aligned} &\frac{d^2x}{dt^2} - 4y = e^t \quad ★3 \\ &\frac{d^2y}{dt^2} + t^2\frac{dx}{dt} = \sin t \quad ★3b \end{aligned}}$$

as a 1st order system

Introduce new variables x_2, y_2

Let $x_2 = \frac{dx}{dt}$

$y_2 = \frac{dy}{dt}$

$$\frac{dx}{dt} = x_2$$

$$\frac{dx_2}{dt} = 4y + e^t \quad ★3$$

$$\frac{dy}{dt} = y_2$$

$$\frac{dy_2}{dt} = -t^2x_2 - \sin t$$

A first order system in 4 variables x, x_2, y, y_2

$$Dx_1 = -3Ae^{-3t} + 2Be^{2t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} Ae^{-3t} + Be^{2t} \\ -2Ae^{-3t} + \frac{B}{2}e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} A + B \\ -2A + \frac{B}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow B = 4A \quad 5A = 1 \quad \Rightarrow A = \frac{1}{5} \quad B = \frac{4}{5}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{5}e^{-3t} + \frac{4}{5}e^{2t} \\ -\frac{2}{5}e^{-3t} + \frac{2}{5}e^{2t} \end{pmatrix}$$

Took a system and wrote it as a 2nd order DE. Then used the roots of the aux poly to solve it

$$\begin{pmatrix} x \\ x_2 \\ y \\ y_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -t^2 & 0 & 0 \end{pmatrix}}_{A(t)} \begin{pmatrix} x \\ x_2 \\ y \\ y_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ e^t \\ 0 \\ -\sin t \end{pmatrix}}_{b(t)} \qquad x'(t) = A(t)\,x(t) + b(t)$$

Vector Formulations

$$\vec{x}'(t) = Ax(t) + B(t)$$

Vector space $V_n(I) = \left\{ \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} : x_i : I \to \mathbb{R} \right\}$

(∞ dimensional)

How do you determine if vectors of functions $\{v_1, \dots v_n\}$ in V_n are Linearly Independent?

Wronskian $v_1(t), \dots v_n(t) \in V_n$

$$W(v_1, \dots v_n)(t) = \det(v_1, \dots, v_n)$$

If $W(v_1 \cdots v_n(t)) \neq 0$ for any t then they're independent

Ex: $v_1 = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 3\sin t \\ \cos t \end{pmatrix}$ $\quad v_1, v_2 \in V_2$

$$W(t) = \begin{vmatrix} e^t & 3\sin t \\ 2e^t & \cos t \end{vmatrix}$$

$$= e^t(\cos t - 6\sin t)$$

$W(0) = 1(1 - 0)$

$= 1 \neq 0$

$\Rightarrow v_1, v_2$ independent

Interval $[a, b]$

$V_n(I) = \{$ vectors of functions from I to $\mathbb{R} \}$

V_n is a VECTOR space.

closed under addition, and scalar multiplication

$$k \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} k x_1(t) \\ \vdots \\ k x_n(t) \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} x_1 + y_1(t) \\ \vdots \\ x_n + y_n(t) \end{pmatrix}$$

$$a_1 v_1 + a_2 v_2 + \cdots a_n v_n = 0$$

$\exists\, a_1, a_2, \dots, a_n$ not all 0 then $v_1, \dots v_n$ are dependent.

But if $\exists\, t_0$ st $a_1 v_1(t) + \cdots + a_n v_n(t) = \vec{0}$

$$\begin{bmatrix} v_{11}(t) & v_{21}(t) & v_{n1}(t) \\ \vdots & & \\ v_{n1}(t) & v_{n2}(t) & v_{nn}(t) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \vec{0}$$

Thm: $v_1 \dots v_n \in V_n(I)$ are lin indep if $\exists\, t \in I$ st $W(v_1 \dots v_n)(t) \neq 0$

Ex: In the above if $I = \mathbb{R}$

$W(0) = e^0 \cos 0 = 1 \neq 0$.

Remark: $\dim(V_n) = +\infty$.

Thm:

Given $\underbrace{x'(t) = A(t)x(t)}_{\text{linear homogeneous system}}$ of continuous fns if $A(t)$ is an $n\times n$ matrix then the solution space is an n dimensional. subspace of V_n

Existence + Uniqueness Theorem.

Ex: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$x_1 = Ae^{-3t} + Be^{2t}$$
$$x_2 = -2Ae^{-3t} + \frac{B}{2}e^{2t} = A\begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix} + B\begin{pmatrix} e^{2t} \\ \frac{1}{2}e^{2t} \end{pmatrix}$$

$\cos t \quad \sin t \in V_1(t)$. we wanted to show

$\cos t$ & $\sin t$ Linearly indep.

2 in V_1 $\times$

$$W(t) = \det\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \cos^2 + \sin^2 = 1 \neq 0$$

$$= \text{span}\left\{ \overbrace{\begin{pmatrix} e^{-3t} \\ -2e^{-3t} \end{pmatrix}}^{\in V_2}, \overbrace{\begin{pmatrix} e^{2t} \\ \frac{1}{2}e^{2t} \end{pmatrix}}^{\in V_2} \right\}$$

2 vectors

SOLUTION space is 2 dimensional

Again the way to PROVE this is using a uniqueness theorem.

Thm: If $x_1, \dots x_n$ are lin. indep. solutions to $x' = Ax$ and A is continuous $n \times n$, then

$$W(x_1, \dots, x_n)(t) \neq 0 \text{ for any } t \in I$$

(Cor: If $x_1, \dots, x_n$ solutions and $W(x_1 \cdots x_n) = 0$ for some t then They are lin. dependent)

Ex: $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ Solve $x' = Ax$

Find evectors and evalues of A.

$P(\lambda) = (\lambda - 1)^2 + 4 = 0 \Rightarrow (\lambda - 1) = \pm\sqrt{-4}$

$\Rightarrow \lambda = 1 \pm 2i$

$$W(t) = \det \begin{vmatrix} e^{-3t} & e^{2t} \\ -2e^{-3t} & \frac{1}{2}e^{2t} \end{vmatrix} = \frac{1}{2}e^{-t} + 2e^{-t} = \frac{5}{2}e^{-t} \neq 0$$

THIS IS A DIFFERENT WRONSKIAN BUT THE IDEA IS VERY SIMILAR

Find n linearly independent solutions to $x' = Ax$. So you have candidates $x_1, \dots x_n$ that all satisfy $x_i' = Ax_i$

$W(x_1 \cdots x_n)(t) = 0 \Rightarrow x_1, \dots x_n$ are not linearly indep.

able to find some

2nd method to solve linear systems of DEs with constant coefficients.

A, has eval λ, and v $\quad Av = \lambda v$

Then $e^{\lambda t} v$ is a solution.

$x = e^{\lambda t} v \quad x' = \lambda e^{\lambda t} v \stackrel{?}{=} Ax = A(e^{\lambda t} v)$

$= e^{\lambda t} A v = \lambda e^{\lambda t} v$ ✓

$$\lambda = \overset{1+2i}{a+ib}$$

$$(A-\lambda I)v = 0 \qquad (Av = \lambda v)$$

$$\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$-2i v_1 + \underset{\uparrow \text{ free variable}}{2v_2} = 0 \qquad v_2 = 1 \Rightarrow v_1 = \frac{\cancel{2}}{\cancel{2}i}\cdot\frac{(-i)}{(-i)}$$

$e^{\lambda t} v$ is a solution and so is $e^{\bar{\lambda} t}\bar{v}$

Lets check that they're linearly independent

$$\begin{bmatrix} e^{\lambda t} & e^{\bar\lambda t} \\ i e^{\lambda t} & -i e^{\bar\lambda t} \end{bmatrix} = -2i\, e^{(\lambda+\bar\lambda)t}$$

$$= -2i\, e^{2t} \neq 0$$

Verify also that the REAL solutions

$$x_1 = \begin{bmatrix} e^t \cos 2t \\ e^t \sin 2t \end{bmatrix} \qquad x_2 = \begin{bmatrix} -e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$$

$$x(t) = A\begin{bmatrix} -e^t \sin 2t \\ e^t \cos 2t \end{bmatrix} + B\begin{bmatrix} e^t \cos 2t \\ e^t \sin 2t \end{bmatrix}$$

$$\det(A-\lambda I) = \underbrace{\begin{vmatrix} 1-\overset{1+2i}{\lambda} & 2 \\ -2 & 1-\lambda \end{vmatrix}}_{A-\lambda I} = P(\lambda)$$

$e^{1+2i} v_1, \quad e^{1-2i} v_2$ Find these 2 eigenvectors

$$= \frac{-i}{-i^2} = \frac{-i}{1} = -i$$

$$v = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Fact $e^{1+2i} v$ is a solution

$\Rightarrow \overline{e^{1+2i} v} = e^{1-2i}\bar{v}$ is also a solution

λ has eigenvector v

$\bar\lambda$ has eigenvector $\bar v$

$$\underbrace{e^{(1+2i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}}_{\text{linearly indep}}, \quad \underbrace{e^{(1-2i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}}_{\text{linearly indep}}$$

If linearly indep

$\Rightarrow$ We have found a basis for solutions.

$$x(t) = A e^{(1+2i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix} + B e^{(1-2i)t}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

are also linearly indep.

complex solutions:

$$e^{(1+2i)t}\begin{bmatrix}1\\ i\end{bmatrix} = \begin{bmatrix} e^t(\cos 2t + i\sin 2t)\\ e^t(i\cos 2t - \sin 2t)\end{bmatrix}$$

$$= \begin{bmatrix} e^t\cos 2t\\ -e^t\sin 2t\end{bmatrix} + i\begin{bmatrix} e^t\sin 2t\\ e^t\cos 2t\end{bmatrix}$$

$$e^{\bar{\lambda} t}\bar{v} = \begin{bmatrix} e^t\cos 2t\\ -e^t\sin 2t\end{bmatrix} - i\begin{bmatrix} e^t\sin 2t\\ e^t\cos 2t\end{bmatrix}$$

Thus since solutions form a vectorspace

$$\begin{bmatrix} e^t\cos 2t\\ -e^t\sin 2t\end{bmatrix} \qquad \begin{bmatrix} e^t\sin 2t\\ e^t\cos 2t\end{bmatrix}$$

are the REAL solutions

$$e^{(1+2i)t}\begin{bmatrix}i\\ 1\end{bmatrix} = \begin{bmatrix} e^t e^{2it}\, i\\ e^t e^{2it}\end{bmatrix}$$

$$= \begin{bmatrix} e^t(\cos 2t + i\sin 2t)\, i\\ e^t(\cos 2t + i\sin 2t)\end{bmatrix}$$

$$= \begin{bmatrix} -e^t\sin 2t & + & i e^t\cos 2t\\ e^t\cos 2t & + & i e^t\sin 2t\end{bmatrix}$$

$$e^{\lambda t}v = \begin{bmatrix} -e^t\sin 2t\\ e^t\cos 2t\end{bmatrix} + i\begin{bmatrix} e^t\cos 2t\\ e^t\sin 2t\end{bmatrix}$$

$$\overline{e^{\lambda t}v} = \begin{bmatrix} -e^t\sin 2t\\ e^t\cos 2t\end{bmatrix} - i\begin{bmatrix} e^t\cos 2t\\ e^t\sin 2t\end{bmatrix}$$

1) $x' = \underset{2\times 2}{A} x$

2) Find eigen values λ_1, λ_2 with evectors v_1, v_2

3) General solution $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$