lec26: Ex:

$$x_1' = 2x_1 + x_2$$
$$x_2' = -3x_1 - 2x_2$$

$\Updownarrow$

$x' = Ax$ where $A = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ -3 & -2-\lambda \end{vmatrix}$$

$$P(\lambda) = (2-\lambda)(-2-\lambda) - (-3)$$
$$= (\lambda-2)(\lambda+2) + 3 = \lambda^2 - 4 + 3 = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$$\lambda = \pm 1$$

$e^{t} v_1 \qquad e^{-t} v_{-1}$

$(A - \lambda I)v = 0$

$E_{-1}$: $\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}$

$3v_1 + v_2 = 0$ $\quad$ Set $v_1 = 1 \Rightarrow v_2 = -3$

$-3v_1 - v_2 = 0$

$\begin{bmatrix} 1 \\ -3 \end{bmatrix} = v_{-1}$

$E_1$: $\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$(A - I)v = 0$

$v_1 + v_2 = 0$

$-3v_1 - 3v_2 = 0$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (Choose $v_1 = 1$)

So our general solution is

$$A e^{-t} \overbrace{\begin{bmatrix} 1 \\ -3 \end{bmatrix}}^{v_{-1}} + B e^{t} \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}^{v_1} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$x_1(0) = 7$

$x_2(0) = 9$

$$x_1(t) = A e^{-t} + B e^{t}$$

$$x_2(t) = -3A e^{-t} - B e^{t}$$

$$7 = A e^{-0} + B e^{0}$$

$$9 = -3A - B$$

$\rightarrow$

$$A + B = 7$$

$$-3A - B = 9$$

Ex: Find the solution to

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 4 & -3 \\ -2 & 2 & -1 \end{bmatrix}_{3\times 3} \qquad x' = Ax$$

(Eigenvalues are $-1, 2$) $\downarrow$ $m_2 = 2$

(will solve in class)

$$P(\lambda) = \det(A - \lambda I)$$

$$= \begin{vmatrix} -\lambda & 2 & -3 \\ -2 & 4-\lambda & -3 \\ -2 & 2 & -1-\lambda \end{vmatrix}$$

$$-\lambda\left[(4-\lambda)(-1-\lambda) - (-6)\right] - (-2)\left[2(-1-\lambda) - (-6)\right] - 2\left[-6 - (-3)(4-\lambda)\right]$$

$$= -\lambda\left[\lambda^2 - 3\lambda - 4 + 6\right] + 2\left[-2 - 2\lambda + 6\right] - 2\left[-6 + 12 - 3\lambda\right]$$

$$= -\lambda\left[\lambda^2 - 3\lambda + 2\right] + 2\left[4 - 2\lambda + 3\lambda - 6\right]$$

$$= -\lambda\left[\lambda^2 - 3\lambda + 2\right] + 2\left[\lambda - 2\right] \qquad = -\lambda\left[(\lambda-2)(\lambda-1)\right] + 2\left[\lambda - 2\right]$$

$$= (\lambda-2)\left[-\lambda(\lambda-1) + 2\right] \qquad = (\lambda-2)\left[-\lambda^2 + \lambda + 2\right] = -(\lambda-2)\left[\lambda^2 - \lambda - 2\right]$$

$$= -(\lambda-2)(\lambda-2)(\lambda+1) \qquad = -(\lambda-2)^2(\lambda+1)$$

$\lambda = 2 \qquad m_2 = 2$

$\lambda = -1 \qquad m_{-1} = 1$

$$E_2 = (A-2I)v = 0 \qquad (A-2I) = \begin{bmatrix} -2 & 2 & -3 \\ -2 & 2 & -3 \\ -2 & 2 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} -2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_2 = t, \; v_3 = s$$

$$-2v_1 + 2v_2 - 3v_3 = 0 \Rightarrow v_1 = v_2 - \frac{3}{2}v_3 = t - \frac{3}{2}s$$

$$S = \left\{ \left(t - \frac{3}{2}s, t, s\right) \right\}$$

linearly INDEP eigenvectors corresponding to eigenvalue 2.

$t=1, s=0 \qquad (1,1,0)$

$t=0, s=1 \qquad (-\frac{3}{2}, 0, 1)$

$$E_2 = \text{span}\left\{ (1,1,0), \left(-\frac{3}{2}, 0, 1\right) \right\}$$

$$E_{-1} = \ker(A+I) \qquad A+I = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 5 & -3 \\ -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 9 & -9 \\ 0 & 6 & -6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad v_3 = t \qquad v_2 = t$$

$$v_1 + 2v_2 - 3v_3 = 0 \Rightarrow v_1 = t$$

$$S = \{(t, t, t)\} = \{t(1,1,1)\}$$

$$\dim(E_2) = m_2 = 2 \qquad \dim(E_{-1}) = 1 = m_{-1}$$

$$\vec{x}(t) = A e^{2t} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + B e^{2t} \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} + C e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Ex: $\begin{bmatrix} 2 & -1 \\ 2 & 4 \end{bmatrix}$ complex eigenvalues.

Solve $x' = Ax$.

$(2-\lambda)(4-\lambda) + 2 = 0$ $\lambda^2 - 6\lambda + 10 = 0$

$(\lambda-3)^2 - 9 + 10 = 1$ $\lambda = 3 \pm i$

Lets find the eigenvectors.

$\begin{bmatrix} 2-3-i & -1 \\ 2 & 4-3-i \end{bmatrix}$ It's clear that there is effectively just one equation

$(-1-i)v_1 - v_2 = 0$ so if $v_2 = 1$

$v_1 = \dfrac{1}{-1-i} \times \dfrac{(-1+i)}{(-1+i)} = \dfrac{-1+i}{1+1}$

$\begin{bmatrix} \frac{-1+i}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \vec{u} + i\vec{v}$

So now we have our 2 "real eigenvectors"

We know $e^{\lambda t}(\vec{u}+i\vec{v})$ is a solution.

If $a+ib=\lambda$

then $e^{at}(\cos bt + i\sin bt)(\vec{u}+i\vec{v})$

is a solution

$= e^{at}(\cos bt\,\vec{u} - \sin bt\,\vec{v})$

$+ i\,e^{at}(\cos bt\,\vec{u} + \sin bt\,\vec{v})$

Therefore (as before 2 linearly indep)

REAL solutions are:

$w_1 = e^{3t}\left(\cos t\begin{bmatrix}-1/2\\1\end{bmatrix} - \sin t\begin{bmatrix}1/2\\0\end{bmatrix}\right)$

$w_2 \quad e^{3t}\left(\cos t\begin{bmatrix}-1/2\\1\end{bmatrix} + \sin bt\begin{bmatrix}1/2\\0\end{bmatrix}\right)$

A general solution is a linear combination

$a w_1 + b w_2(t)$

In summary:

If $A$ is an $n \times n$ matrix of real constants

$\lambda$ is a real eigenvalue with $k$

indep. eigenvectors $v_1, \ldots v_k$

$e^{\lambda t} v_i \quad i = 1, \ldots, k$ are lin. indep

solutions

(Check indep by computing Wronskian)

2) If $\lambda$ is complex then $\exists$ $k$ complex

valued solutions $e^{\lambda t} v_1, \ldots e^{\lambda t} v_k$

and 2k real valued solutions