Quiz question

1) Let X, Y are independent and uniform on $\{1, 2, \ldots, n\}$.

Suppose $Z = X + Y$

1) Find the joint pmf of X and Y
2) Find the range of values Z can take.
3) Find $p_Z(i)$

Solution:

$$P_{X,Y}(i,j) = \begin{cases} \frac{1}{n} \cdot \frac{1}{n} & i,j \in \{1,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$$

$$Z \in \{2, \dots, 2n\}$$

$$P_Z(k) = \sum_i P(X=i, Y=k-i) = \sum_i P(X=i) P(Y=k-i)$$

Note $1 \le i \le n$ of course

$1 \le k-i \le n \Rightarrow k-1 \ge i \ge k-n$

$$= \sum_{i=\max(1,k-n)}^{\min(n,k-1)} \frac{1}{n^2}$$

The limits come into play when $k-1 < n \Rightarrow k < n+1$

Or $k \ge n+1$

Then

$$\begin{cases} \frac{k-1}{n^2} & k < n+1 \\ \frac{n-(k-n)+1}{n^2} & k \ge n+1 \end{cases}$$