without replacement

Quiz: Let 2 cards be drawn at random ^ from a pack of N. Each card is labeled $\{1, 2, \cdots, N\}$.

Let X_1 and X_2 be the values of the 1st two

Find

$$E[X_2 \mid X_1 = k] \quad \text{for} \quad k = 1, 2, \cdots, N$$

and then find $E[X_2]$

Solution: To find $E[X_2 | X_1 = k]$, need $p_{X_2|X_1}(i|k)$

$\Omega = \{ (i_1, i_2, \ldots, i_n) : i_p \neq i_q,\ p \neq q \in \{1, 2, \ldots, n\} \}$

$= S_n$ (space of permutations) $|\Omega| = n!$

$$p_{X_2|X_1}(i|k) = \frac{p_{X_2,X_1}(i,k)}{p_{X_1}(k)} = \frac{(n-2)!/n!}{1/n} = \frac{1}{n-1} \quad \text{if } i \neq k$$

You can also directly just state

$$p_{X_2|X_1}(i|k) = \frac{1}{n-1} \qquad i \neq k$$

Then $$E[X_2 | X_1 = k] = \sum_{i=1}^{n} i\, p_{X_2}(i|k) = \sum_{i=1}^{n} \frac{i}{n-1} - \frac{k}{n-1}$$

$$= \frac{n(n+1)}{2(n-1)} - \frac{k}{n-1} = \frac{n(n+1) - 2k}{2(n-1)}$$

Extra: $$E[X_2] = E\big[E[X_2 | X_1]\big] = \sum_{k=1}^{n} \frac{1}{n} \cdot \left(\frac{n(n+1)}{2(n-1)} - \frac{2k}{2(n-1)} \right)$$

$$= \frac{1}{n} \left(\frac{n^2(n+1) - n(n+1)}{2(n-1)} \right) = \frac{n(n+1)}{2}$$

Directly: $$E[X_2] = \sum_{k=1}^{n} k\, p_{X_2}(k) = \sum_{k=1}^{n} k \cdot \frac{1}{n} = \frac{n(n+1)}{2}$$